Display
D

UNIVERSITY
OF MICHIGAN
MAY 27 1954
MATH. ECON.
LIBRARY

Mathematical Reviews

Edited by

R. P. Boas, Jr. E. Hille H. Whitney

J. V. Wehausen, *Executive Editor*

Vol. 15, No. 4 April, 1954 pp. 275-382

TWO PARTS—PART ONE

TABLE OF CONTENTS

History 275
Foundations 277
Algebra 279
Abstract algebra 279
Theory of groups 283
Number theory 288
Analysis 293
Theory of sets, theory of functions of real variables 296
Theory of functions of complex variables . 300
Theory of series 304
Fourier series and generalizations, integral transforms 306
Polynomials, polynomial approximations . 308
Harmonic functions, potential theory . . 309
Differential equations 311
Difference equations, special functional equations 323
Functional analysis, ergodic theory . . . 324
Calculus of variations 328
Theory of probability 329
Mathematical statistics 330
Mathematical economics 333
Topology 334
Geometry 339
Convex domains, extremal problems, integral geometry 341
Algebraic geometry 342
Differential geometry 345
Numerical and graphical methods . . . 352
Astronomy 355
Relativity 357
Mechanics 359
Hydrodynamics, aerodynamics, acoustics . 360
Elasticity, plasticity 369
Mathematical physics 373
Optics, electromagnetic theory 374
Quantum mechanics 378

AUTHOR INDEX

Acsél, J. 324
Adams, G. 339
Adkins, J. E.-Green, A. E.-Shield, R. T. 369
Aeschlimann, F. 382
Agnew, R. P. 305
Agostini, A. 276
*Aleksandrov, P. S. 296
Alguneid, A. R. 343
Amemiya, A. See Kotani, M.
Andreotti, A. 344
Antonowicz, K. 355
Apéry, R. 294
Archibald, R. C. 276
Arelkin, G. Ya. 334
Arin'kš, E. G. 297
Arsove, M. G. 376
Artzy, R. 346
Arľanyh, I. S. 317, 360
Ascari, A. 312
Ascoli, G. 312
Atiya, F. S. 377
Aymerich, G. 313
Baer, R. 285
Bagemihl, F.-Seidel, W. 295
Bailey, W. N. 289
Bambah, R. P. 292
Banachiewicz, T. 354
Barnard, G. A. 332
Barner, M. 349
Bartlett, M. S.-Rajalakshman, D. V. 333
Basilevich, V. 372
Bayard, M. 377
Beckert, H. 310
Beckman, F. S.-Quarles, D. A., Jr. 335
Bellin, A. I. 313
Bellman, R. 330
Benson, G. C. See van der Hoff, B. M. E.
Berezina, L. Ya. 348, 374
Berger, W. J. See Saibel, E.
Bergmann, H. G. 314
Bers, L. 304
Bertolini, F. 329
Beth, E. W. 277
Beurling, A.-Helson, H. 307
*Bezuhov, N. I. 369
Bhatia, A. B.-Wolf, E. 308
*Bianchi, L. 276
Biernacki, M. 294
Bilo, J. 339
Birman, M. S. 326
Birnbaum, A. 331
Blackett, D. W. 281
*Blaschke, W. 275
Blaschke, W. 346
Bleksley, A. E. H. 357
Block, H. D. 345
Bochner, S. 295
*Bohr, H. 276
Bondar', N. G. 372
Bonnor, W. B. 358
Bonsall, F. F. 325
Borel, A.-Serre, J.-P. 338
Borell, M. 364
Borg, G. 315
Borsuk, K. 337
Bosanquet, L. S. 307
Bothwell, F. E. 333
Bouligand, G. 279, 345
Box, G. E. P. 332
Brachman, M. K. See Hellund, E. J.
Browder, F. 320
Brueckner, K. A.-Watson, K. M. 379
Brun, V. 276
*Bullen, K. E. 373
Burgers, J. M. 366
Busbridge, I. W. 357
Butzer, P. L. 309
Buzano, P. 346
Buzin, E. I. 371
Cañero, F. 297
Calabi, E.-Rosenlicht, M. 351
Calderon, A. P.-Mann, H. B. 298
Carlitz, L. 289, 291, 308, 324
Cassels, J. W. S. 293
Cassina, U. 276
Castoldi, L. 360, 378
Cattaneo, C. 359
Černikov, S. N. 293, 294
Cerrillo, M. V.-Guillemin, E. A. 376
Chamberlin, E.-Wolf, J. 339
Charnes, A.-Lemke, C. E. 354
Chern, Shiing-shen 347
Chraplyvy, Z. V. 382
Chu, Sheng To. See Tifford, A. N.
Chuang, Chi-Tai 295
de la Cinta Badillo, M. 276
Clagett, M. 275, 276, 277
Clement, P. A. 347
Clement, P. P. 309
Cohen, H. G. 313
Cole, J. D.-Dougherty, C. B.-Huth, J. H. 373
Consiglio, A. 359
Conte, L. 276
Conte, S. D.-Sangren, W. C. 314
Cooper, R. 306
Corinaldesi, E. 378
van der Corput, J. G. 355
Costa de Beauregard, O. 381
Cox, D. R. 332
Cramér, H. 276
Crespo Pereira, R. 277
Croisot, R. See Dubreil-Jacotin, M. L.
Čudakov, N. G. 289
Cunihin, S. A. 286
Curtiss, C. F. 381
Danskin, J. M.-Gillman, L. 333
Darbo, G. 298
Darlington, S. 377
David, F. N. 354
Davis, P. 295
De Cicco, J. See Kasner, E.
Decuyper, M. 348
Dedò, M. 343
Delavault, H. 323
Destouches-Février, P. 382
Díaz, J. B.-Weinberger, H. F. 321
Dienst, H.-R. 353
Dieudonné, J. 287
Dinghas, A. 303, 309
Dirac, P. A. M. 378
Dmitriev, A. A. 363
Dobrovol'skiĭ, V. V. 359
Dobrušin, R. L. 329
Dodd, K. N. 367
Doss, S.-Nasr, S. K. 324
Dougherty, C. B. See Cole, J. D.
Downton, F. 331
Drazin, M. P. 282
Drazin, M. P.-Gruenberg, K. W. 282
Dresden, A. 279
Drukarev, G. F. 381
*Dubnov, Ya. S. 339
Dubošin, G. N. 355
Dubovickiĭ, A. Ya. 299
*Dubreil-Jacotin, M. L.-Lesieur, L.-Croisot, R. 279
Dugas, R. 276
Duparc, H. J. A.-Peremans, W. 288
Durand, E. 382
Dvoretzky, A.-Kiefer, J.-Wolfowitz, J. 333
Dwyer, P. S. 354
Dyer, E. 335
Dynkin, E. B. 282
Eder, G. 282
Efgrafov, M. A. 323
Èldus, D. M. 319
Einstein, A. 357
Elenevskaya, N. B. 355
Ellis, D. 287
Erim, K. 279
van Est, W. T. 288
Estermann, T. 292
Evans, T. 283
Evans, T.-Neumann, B. H. 284
*Euclid 276
Fabre de la Ripelle, M. 389
Faure, R. 327
Feit, W. 287
Fejes Tóth, L. 341

(*Continued on cover 3*)

UNESCO BOOK COUPONS

Unesco Book Coupons may be sent directly to the American Mathematical Society in payment for its publications. A folder describing these coupons and listing the countries in which they are available may be obtained from the Society upon request.

MATHEMATICAL REVIEWS

Published monthly, except August, by

THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania

Sponsored by

THE AMERICAN MATHEMATICAL SOCIETY
THE MATHEMATICAL ASSOCIATION OF AMERICA
THE INSTITUTE OF MATHEMATICAL STATISTICS
THE EDINBURGH MATHEMATICAL SOCIETY
SOCIÉTÉ MATHÉMATIQUE DE FRANCE
DANSK MATEMATISK FORENING
HET WISKUNDIG GENOOTSCHAP TE AMSTERDAM
THE LONDON MATHEMATICAL SOCIETY
POLSKIE TOWARZYSTWO MATEMATYCZNE
UNIÓN MATEMÁTICA ARGENTINA
INDIAN MATHEMATICAL SOCIETY
UNIONE MATEMATICA ITALIANA

Editorial Office

MATHEMATICAL REVIEWS, 80 Waterman St., Providence 6, R. I.

Subscriptions: Price $20 per year ($10 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions may be addressed to MATHEMATICAL REVIEWS, Lancaster, Pennsylvania, but should preferably be addressed to the American Mathematical Society, 80 Waterman St., Providence 6, R. I.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. Its preparation is also supported currently under a contract with the Office of Scientific Research, Headquarters, Air Research and Development Command, U. S. Air Force. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in paragraph (d-2), section 34.40, P. L. and R. of 1948, authorized November 9, 1940.

Mathematical Reviews

Vol. 15, No. 4 (Two Parts—Part One) APRIL, 1954 Pages 275-382

HISTORY

*Hofmann, Joseph Ehrenfried. **Geschichte der Mathematik. Erster Teil. Von den Anfängen bis zum Auftreten von Fermat und Descartes.** Sammlung Göschen Bd 226. Walter de Gruyter & Co., Berlin, 1953. 200 pp. DM 2.40.

This extremely compact history of mathematics from its origins in prehistoric days to the great reconstruction of the 17th century has many interesting characteristics. It is relatively short on pre-Grecian and Grecian mathematics and places its main emphasis on the European medieval and the Renaissance periods. The reason is that there already exist good reports on antique mathematics, among them the first section of the recent book by O. Becker and the author [Geschichte der Mathematik, Athenäum, Bonn, 1951; these Rev. 14, 341], but there exist no comparable histories of the other periods. Particular attention is bestowed on the mathematics of the scholastics and prescholastics, beginning with Origenes and ending with Oresme and Bradwardine. Equal care has been paid to the humanistic and "early baroque" periods. And since the author has actually had at least a look (and often much more than that) into almost all the original sources and has studied the best available secondary sources he has been able to give a fresh and often a novel approach to the works of the mathematicians of the past. The booklet abounds in little original touches, and tries to avoid those classical pitfalls of historians which consist in carelessly repeating the errors of their predecessors. The author has been extremely generous in his references to source and secondary material—the index of names, consisting of 43 pages, is a minor encyclopedia—and has tried with success to be accurate in detail. An outstanding aspect of the exposition is also that its subject is not treated in isolation, but as an integral part of the general culture of its period. Historians of related fields will find many useful remarks in this discussion of mathematics.

Because of its many details in names and facts, which naturally had to be sketchy in view of the size of the book, this work will be most useful to persons who already possess some preparatory knowledge or who like quick and reliable reference, even on obscure authors. The neophyte will find the details rather overwhelming. However, the first impression, that the author wants to drown the gentle reader in an ocean of general erudition, is not borne out by further study; certain main characters stand firmly out and are given a shrewd appraisal of their merits. Examples are Regiomontanus, Stevin and Viète. Outstanding among many other excellent detail studies is also the history of mathematics among the Jesuits. *D. J. Struik.*

*Blaschke, Wilhelm. **Griechische und anschauliche Geometrie.** Verlag von R. Oldenbourg, München, 1953. 60 pp.

This booklet (no. 1 of the "Mathematische Einzelschriften" published under the direction of the author) contains lectures given during the summer semester 1952 at Hamburg University as part of the "Studium Generale" and is aimed at a larger public than the professional mathematicians. The author takes the principal Greek mathematicians of Antiquity from Pythagoras to Pappos and shows how some of their problems have influenced later work or are still affecting our present research. For instance, Euclid's discussion of the Platonic bodies leads to convexity considerations and some of Pappos' geometrical demonstrations to a discussion of the theorem of Descartes and Pascal. There are sections on Euler's polyhedron formula, the construction of polyhedra by paper folding, Archimedes' quadrature of the parabola, the axiom of Archimedes, Zenodoros' isoperimetrical problems and their influence on Steiner, Schwarz and Minkowski. The lectures end with the trisection of the angle. *D. J. Struik.*

Gillings, R. J. Unexplained error in Babylonian cuneiform tablet, Plimpton 322. Australian J. Sci. 16, 54–56 (1953).

Clagett, Marshall. The medieval Latin translations from the Arabic of the Elements of Euclid, with special emphasis on the versions of Adelard of Bath. Isis 44, 16–42 (1953).

*Geronimus, Ya. L. **Očerki o rabotah korifeev russkoĭ mehaniki. [Essays on the work of the leaders of Russian mechanics.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1952. 519 pp. (13 plates). 22.75 rubles.

The following Russian scientists are discussed in the collection of essays: M. V. Ostrogradskiĭ (1801–1862); O. I. Somov (1815–1876); P. L. Čebyšev (1821–1894); I. A. Vyšnegradskiĭ (1831–1895); V. G. Imšeneckiĭ (1832–1892); S. V. Kovalevskaya (1850–1891); N. E. Žukovskiĭ (1847–1921); S. A. Čaplygin (1869–1942); A. M. Lyapunov (1857–1918); V. A. Steklov (1864–1926); I. V. Meščerskiĭ (1859–1935); and A. N. Krylov (1863–1945). The individual essays are planned as follows: a short biography of the person treated; an exposition of his most important work; a discussion of the general characteristics of his scientific work and its later development by and its effect on other Russian scientists.

Seydl, Otto. The history of the "Museum Mathematicum Jesuitarum" in the college of St. Clemens in the old town of Prague. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. 1951, no. 7, 59 pp. (1953). (Czech. English summary)

*Gloden, A. **Liste des travaux d'histoire des sciences et de la technique dus à des luxembourgeois de 1839 à 1953.** Ministère de l'Education Nationale du Grand-Duché de Luxembourg, 1953. 11 pp.

Picone, Mauro. On the mathematical work of the Italian Institute for the Application of Calculus during the first quarter century of its existence. Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. 362, 37 pp. (1953).

Hua, Lo-Kèn. The present position of mathematics in China. Vestnik Akad. Nauk SSSR 6, 14–20 (1953). (Russian)

Lange-Nielsen, Fr. Niels Henrik Abel. Nordisk Mat. Tidskr. 1, 65–90, 143 (1 plate) (1953). (Norwegian. English summary)

Brun, Viggo. The manuscript of Abel's Paris treatise found. Nordisk Mat. Tidskr. 1, 91–97, 143–144 (1 plate) (1953). (Norwegian. English summary)

Clagett, Marshall. A medieval fragment of the De sphaera et cylindro of Archimedes. Isis 43, 36–38 (1952).

de la Cinta Badillo, María. Jacobo Bernoulli. Gaceta Mat. (1) 5, 103–105 (1 plate) (1953). (Spanish)

Procissi, Angiolo. Gli studi di Enrico Betti sulla teoria di Galois nella corrispondenza Betti-Libri. Boll. Un. Mat. Ital. (3) 8, 315–328 (1953).

***Bianchi, Luigi. Opere. Vol. I, Parte seconda.** A cura dell'Unione Matematica Italiana e col contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese della Casa Editrice Perrella, Roma, 1953. 276 pp. 2500 lire.

This completes vol. I of the projected 10-volume edition of Bianchi's work [cf. these Rev. 14, 524]. This part contains his papers on normal elliptic curves, finite continuous groups of transformations, partial differential equations, and, in addition, various obituaries which he wrote. There are introductory comments to the first three groups of papers by F. Conforto, U. Amaldi, and E. Bompiani and Enea Bortolotti, respectively.

***Bohr, Harald. Collected mathematical works. Vol. I. Dirichlet series. The Riemann zeta-function. Vol. II. Almost periodic functions. Vol. III. Almost periodic functions (continued). Linear congruences. Diophantine approximations. Function theory. Addition of convex curves. Other papers. Encyclopædia article. Supplements.** Dansk Matematisk Forening, København, 1952. xxxiv+771 pp. (1 plate); ix+852 pp. (1 plate); x+985 pp. (1 plate). 110 kr.

The Supplements in Vol. III contain an English translation of Bohr's thesis, English summaries of his other papers in Danish, Jessen's memorial address [Acta Math. 86, i–xiv (1951); these Rev. 13, 420], a chronological list of Bohr's publications, notes on individual papers, and a general index to the three volumes. The papers are reproduced from the originals by a photographic method.

Frostman, Otto. Fritz Carlson in memoriam. Acta Math. 90, ix–xii (1953).

A list of Carlson's mathematical works is included. Carlson was born 23 July 1888 and died 28 November 1952.

Terracini, Alessandro. Guido Castelnuovo. 1865–1952. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 86, 366–377 (1952).

Ullrich, Egon. Friedrich Engel. Ein Nachruf. Mitt. Math. Sem. Giessen no. 40, 16 pp. (1 plate) (1951).

***Euclid's Elements, Book III.** Translated and annotated by Anton Bilimović. Srpska Akademija Nauka. Klasični Naučni Spisi, Kn. III. Matematički Institut, Kn. 3. Belgrade, 1953. 48 pp. (Serbo-Croatian)

***Euclid's Elements, Book IV.** Translated and annotated by Anton Bilimović. Srpska Akademija Nauka. Klasični Naučni Spisi, Kn. IV. Belgrade, 1953. 31 pp. (Serbo-Croatian)

Agostini, Amedeo. Leonardo Fibonacci. Archimede 5, 205–206 (1953).

Whittaker, Edmund. G. F. FitzGerald. Scientific American 185, no. 5, 93–98 (1953).

Agostini, Amedeo. L'Aritmetica di Galigai Francesco. Period. Mat. (4) 31, 201–206 (1953).

Galli, M. Le prove meccaniche della rotazione terrestre secondo Galileo. Boll. Un. Mat. Ital. (3) 8, 328–336 (1953).

Lorey, Wilhelm. Karl Friedrich Gauss. Zur 175. Wiederkehr seines Geburtstages. Math.-Phys. Semesterber. 3, 179–192 (1953).

Cassina, Ugo. Alcune lettere e documenti inediti sul trattato di calcolo di Genocchi-Peano. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 16(85), 337–362 (1952).

Mandelbrojt, S. The mathematical work of Jacques Hadamard. Amer. Math. Monthly 60, 599–604 (1953).

Koyré, Alexandre. An unpublished letter of Robert Hooke to Isaac Newton. Isis 43, 312–337 (1952).

Dugas, René. Huygens devant le système du monde, entre Descartes et Newton. C. R. Acad. Sci. Paris 237, 1477–1478 (1953).

Conte, Luigi. Huygens e l'invenzione di due medie proporzionali. Period. Mat. (4) 31, 145–157 (1953).

Archibald, R. C. Notes on Omar Khayyām (1050–1122) and recent discoveries. Pi Mu Epsilon J. 1, 350–358 (1953).

Krall, Giulio. Tullio Levi-Cività nella meccanica del suo tempo. Civiltà delle Macchine 1, no. 4, 33–37 (1953).

Krall, Giulio. Tullio Levi-Cività e la relatività. Civiltà delle Macchine 1, no. 6, 42–48 (1953).

Kibre, Pearl. Lewis of Caerleon, doctor of medicine, astronomer, and mathematician (d. 1494?). Isis 43, 100–108 (1952).

Tenca, Luigi. I contrasti fra Alessandro Marchetti e Vincenzio Viviani. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 16(85), 293–313 (1952).

Cramér, Harald. Richard von Mises' work in probability and statistics. Ann. Math. Statistics 24, 657–662 (1953). A selected bibliography is included.

Clagett, Marshall. The use of the Moerbeke translations of Archimedes in the works of Johannes de Muris. Isis 43, 236–242 (1952).

***Gnedenko, B. V. Mihail Vasil'evič Ostrogradskiĭ. Očerki žizni, naučnogo tvorčestva i pedagogičeskoĭ deyatel'nosti [Mihail Vasil'evič Ostrogradskiĭ. Outlines of his life, scientific work and pedagogical activity.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1952. 331 pp. (3 plates). 8.40 rubles.

***Œuvres de Henri Poincaré.** Publiées sous les auspices de l'Académie des Sciences par la Section de Géométrie. Tome VIII. Publié avec la collaboration de Pierre Sémirot. Gauthier-Villars, Paris, 1952. 693 pp. 9000 fr.

This volume completes the publication of Poincaré's papers on astronomy and celestial mechanics started in vol. VII [cf. these Rev. 13, 810]. The papers are grouped as follows: perturbation function and periods of double integrals; the figure of the Earth, theory of tides; lunar theory; theory of planets; mechanical quadratures; cosmogonical hypotheses; other articles, reports and lectures. As in earlier volumes, Poincaré's own analysis of his work is included [Acta Math. 38, 3–135 (1921), pp. 110, 114–115]. There are notes (pp. 681–688) by Sémirot.

Masotti, Arnaldo. Sull'opera scientifica di Matteo Ricci. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 16(85), 415–445 (1 plate) (1952).

Crespo Pereira, Ramón. Thomas Vicente Tosca. Gaceta Mat. (1) 5, 53–60 (1 plate) (1953). (Spanish)

***Wiener, Norbert. Ex-prodigy. My childhood and youth.** Simon and Schuster, New York, 1953. xii+309 pp.

FOUNDATIONS

***Beth, E. W. Sur le parallélisme logico-mathématique.** Les méthodes formelles en axiomatique. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 36, Paris, 1950, pp. 27–32; discussion, pp. 32–33. Centre National de la Recherche Scientifique, Paris, 1953.

This paper contains a very good account of the recent developments and trends toward use of metamathematical methods in the proofs of mathematical theorems, especially in algebra, as in the work of Tarski, Henkin and Robinson.

I. Novak Gál (Ithaca, N. Y.).

Henkin, Leon. Banishing the rule of substitution for functional variables. J. Symbolic Logic 18, 201–208 (1953).

The author proves the equivalence of two formal systems F^2 and F^{**} for the second order functional calculus, which are extensions of the system F^*, resulting from Church's F^1 [Introduction to mathematical logic, vol. I, Princeton, 1944; these Rev. 6, 29] by applying the axiom schema $(\mathfrak{a})\mathfrak{A} \supset S^{\mathfrak{a}}_{\mathfrak{b}}\mathfrak{A}|$ to function variables as well as to individual variables. F^2 is F^* with the axiom schema $(\mathfrak{c})\mathfrak{A} \supset S_{\mathfrak{B}}{}^{\mathfrak{c}(\mathfrak{a}_1,\cdots,\mathfrak{a}_n)}\mathfrak{A}|$ added, while F^{**} is obtained from F^* by adding the axiom schema $(\exists \mathfrak{c})(\mathfrak{a}_1)\cdots(\mathfrak{a}_n)(\mathfrak{c}(\mathfrak{a}_1,\cdots,\mathfrak{a}_n) \equiv \mathfrak{B})$. Thus F^{**} is a sufficiently strong system for the second-order functional calculus in which the cumbersome rule of substitution for functional variables is avoided.

A. Heyting (Amsterdam).

Hailperin, Theodore. Quantification theory and empty individual-domains. J. Symbolic Logic 18, 197–200 (1953).

Mostowski [same J. 16, 107–111 (1951); these Rev. 13, 3] examines those formulas of quantification theory which are valid on all domains, including the empty domain and gives a complete set of axioms for such a first-order functional calculus. By interpreting $(x)p$, where p contains no variables as the conjunction of p with itself for as many times as there are individuals in the domain and using a system of quantification theory which contains no free variables in theorems, a considerably more elegant system than Mostowski's is obtained and the proof of completeness is simplified.

I. Novak Gál (Ithaca, N. Y.).

Quine, W. V. Two theorems about truth functions. Bol. Soc. Mat. Mexicana 10, 64–70 (1953).

Definitions. A fundamental formula (f.f.) is a conjunction of distinct negated or unnegated propositional variables (its components). The f.f. whose components are $(\neg)^{r_i}A_i$, $1 \leq i \leq n$, has the number $1+\sum_{i=1}^{n} r_i 2^{n-i}$. The f.f. Φ subsumes a f.f. Ψ if each component of Ψ is also a component of Φ. A normal formula (n.f.) is a disjunction of f.f. (its clauses), no two of which are permutations of each other. The length of a f.f. exceeds the number of its components by one, the length of a n.f. is the sum of the lengths of its clauses minus one.

Theorem I. If (i) Φ and Ψ are n.f., (ii) Ψ does not contain negation, (iii) no clause of Ψ subsumes any other clause of Ψ, and (iv) $\Phi \leftrightarrow \Psi$, then the length of Ψ does not exceed the length of Φ.

Theorem II. The n.f. Φ whose clauses are the f.f. with numbers $\leq i < 2^n$ is equivalent to a n.f. Ψ which satisfies conditions (ii) and (iii) of Theorem I (Ψ is a shortest normal equivalent of Φ). [The proof depends on the facts that the disjunction of the first 2^{n-k} f.f. is equivalent to $A_1 \& \cdots \& A_k$ and $A \vee (\neg A \& B) \cdot\leftrightarrow\cdot A \vee B$.]

G. Kreisel.

Rose, Alan. Some self-dual primitive functions for propositional calculi. Math. Ann. 126, 144–148 (1953).

For the m-valued propositional calculus with m even, a single primitive function is given, in terms of which all the connectives of the calculus may be defined. The definition is made by combining the conditional disjunction function previously studied by the author [Math. Ann. 123, 76–78 (1951); these Rev. 12, 790] with a new unary function which is analogous to negation.

For the m-valued calculus with m an odd integer, a weaker result is obtained. It is shown that the system consisting of the conditional disjunction function and two unary functions analogous to negation is functionally complete; every connective is definable in terms of these three. It is also shown that every connective of the Lewis calculus of strict implication S2 and of every stronger system, is definable in terms of a single primitive function.

O. Frink (State College, Pa.).

*Frey, Gerhard. **Bemerkungen zum Problem der mehrwertigen Logiken.** Actes du XIème Congrès International de Philosophie, Bruxelles, 20–26 Août 1953, vol. V, pp. 53–58. North-Holland Publishing Co., Amsterdam; Editions E. Nauwelaerts, Louvain, 1953.

Frey first considers an interpretation of 3-valued logic in terms of the concept of provability and shows that this interpretation is impossible. He then considers many-valued logics from the truth-table point of view and attempts to prove that the Wajsberg-Słupecki axioms [Wajsberg, C. R. Soc. Sci. Lett. Varsovie Cl. III, **24**, 126–148 (1932); Słupecki, ibid. **29**, 9–11 (1936)] for three-valued logic can be translated to give two other formalizations in which the designated truth-values are 3 and 2 respectively. He incorrectly attributes the Wajsberg-Słupecki axioms to Łukasiewicz. In the first case he replaces C in all axioms by NC. In the second case he abbreviates $CpTp$ by Sp and replaces C, N, T by SC, SN, ST respectively. Both results are fallacious as may be seen by considering the axiom $CpCqp$. The translations of this are $NCpNCqp$ and $SCpSCqp$ and if the truth-value of p is 1 both these formulae take the truth-value 1. *A. Rose* (Nottingham).

*Martin, R. M. **On non-translational semantics.** Actes du XIème Congrès International de Philosophie, Bruxelles, 20–26 Août 1953, vol. V, pp. 132–138. North-Holland Publishing Co., Amsterdam; Editions E. Nauwelaerts, Louvain, 1953.

The author sketches how a semantical meta-language over a language L can be constructed, which consists of elementary concatenation theory augmented by the theory of comprehension. Here comprehension is considered as a relation between names; a name is any expression of L which is either a one-place predicate or a one-place abstract. Similar ideas were worked out in more detail by the author in another paper [J. Symbolic Logic **18**, 11–18 (1953); these Rev. **14**, 937]. *A. Heyting* (Amsterdam).

Goodstein, R. L. A problem in recursive function theory. J. Symbolic Logic **18**, 225–232 (1953).

This is a complement to an earlier paper of the author [Proc. London Math. Soc. (2) **52**, 81–106 (1950); these Rev. **12**, 664], which contains the definitions of the notions used here. In that paper Rolle's theorem was proved in the following form: If (A) $f(n, x)$ is relatively differentiable with a relative derivative $f'(x)$, for $a \leqq x \leqq b$, and $f(n, a) = f(n, b) = 0$ relative to n, then (R) there is a recursive function c_k, $a < c_k < b$, and a recursive $R(k)$ such that $f'(n, c_k) = 0$ (k) for $n \geqq R(k)$. However, the stronger assertion (U) that there exist α, β such that $a < \alpha \leqq c_k \leqq \beta < b$, was only proved under the added condition (B): $f(n, x)$ is either relatively variable or relatively constant. Here the author proves that (R) with (U) cannot be proved from (A) alone. This goes by the construction of an example, starting from a function $g(x)$ for which $g(k) = 0$ is provable in recursive number theory for every fixed k, while the formula $g(x) = 0$ with the free variable x is not provable. On the other hand it is shown that, given (A), (B) is not a necessary condition for (U). *A. Heyting* (Amsterdam).

Matsumoto, Kazuo. Sur la structure concernant la logique moderne. J. Osaka Inst. Sci. Tech. Part I. **2**, 67–78 (1950).

Let L be a Boolean algebra with operators $\wedge$, $\sim$, in which an additional operator K is defined, satisfying $p \wedge Kp = 0$, $Kp \vee KKp = 1$, $K(p \vee q) = Kp \wedge Kq$. If we define $\Diamond p = \sim Kp$, the resulting system is equivalent (in an obvious sense) to the Lewis system S5. If we define $Np = \sim K \sim p$, $p \supset q = N(p \wedge Nq)$, the author affirms that the resulting system is equivalent to intuitionistic logic. The latter is false, for $NNp \supset p$ is by his definition the same as $N(NNp \wedge Np)$, which is equal to 1 in L. As the proofs are only sketched, the reviewer was not able to detect the mistake. *A. Heyting* (Amsterdam).

*Gilmore, P. C. **Griss' criticism of the intuitionistic logic and the theory of order.** Actes du XIème Congrès International de Philosophie, Bruxelles, 20–26 Août 1953, vol. V, pp. 98–104. North-Holland Publishing Co., Amsterdam; Editions E. Nauwelaerts, Louvain, 1953.

This is a summary of the author's work on the logic of negationless intuitionistic mathematics [Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 162–174, 175–186 (1953); these Rev. **14**, 1053], illustrated by its application to the theory of order. *A. Heyting.*

Shepherdson, J. C. Inner models for set theory. III. J. Symbolic Logic **18**, 145–167 (1953).

[For parts I–II see same J. **16**, 161–190 (1951); **17**, 225–237 (1952); these Rev. **13**, 522; **14**, 834.] By 'set-theory' is meant the set of axioms A, B, C as formulated by Gödel. Let '$A \in_m B$', '$\mathfrak{M}_m(A)$', '$\mathfrak{Cls}_m(A)$' mean respectively 'A is a member of B (in the sense of the model)', 'A is a set of the model', 'A is a class of the model'. It was proved in an earlier paper that every model of axioms A, B, C which satisfies the conditions

i) $(A_m)(\exists B)(X)(X \in_m A_m . \equiv . X \in B)$,

ii) $(\exists E^*)(\mathfrak{Rel}(E^*).(x)(y)(\langle xy \rangle \in E^* . \equiv . \mathfrak{M}_m(y).x \in_m y))$,

iii) $(X)[\mathfrak{M}_m(X). \supset . \mathfrak{M}(\mathfrak{P}(y \in_m X))]$,

iv) $(A)[\sim\mathfrak{Em}(A). \supset . (\exists y)(y \in A.(z)\sim(z \in_m y.z \in A))]$,

is isomorphic to a complete model, i.e., one which satisfies the additional conditions

v) $(X)(Y)(X \in_m Y. \equiv . \mathfrak{M}_m(X).\mathfrak{Cls}_m(Y).X \in Y)$,

vi) $(A_m)(X)(X \in A_m . \supset . \mathfrak{M}_m(X))$.

A complete model is called normal, if there is a normal propositional function '$\mathfrak{A}(X)$' and a special class S such that $\mathfrak{Cls}_m(X). \equiv . \mathfrak{A}(X)$, $\mathfrak{M}_m(X). = . X \in S$, and

$$X \in_m Y. \equiv . \mathfrak{M}_m(X).\mathfrak{Cls}_m(Y).X \in Y.$$

A complete model is called a proper complete model (PCM), if $(\exists C)(X)(\mathfrak{Cls}_m(X). \equiv . X \in C)$. The statement that there are no proper complete models can be expressed within set-theory and it can be shown that if A, B, C are consistent, then so is the system of axioms A, B, C, '$V = L$', '$\sim(\exists c, s)\mathrm{PCM}(c, s)$', where the last axiom is the formalization of the statement that there is no PCM. The following theorems are then proved: (1) In a universe satisfying A, B, C, D, '$V = L$', '$\sim(\exists c, s)\mathrm{PCM}(c, s)$', any normal complete model for axioms A, B, C has its universal class S equal to V and satisfies also the axioms '$V = L$' and '$\sim(\exists c, s)\mathrm{PCM}(c, s)$'; (2) If K is a proposition of the form '$(Qa_1)\cdots(Qa_r)\mathfrak{A}$' where '$\mathfrak{A}$' is a quantifier-free formula built up from contexts of the form '$\mathfrak{M}(a_i)$', '$a_i \in a_j$', and $a_1, \cdots, a_r$ are all the class or set variables in '$\mathfrak{A}$' and '(Qa_i)' is either '(a_i)' or '$(\exists a_i)$' in case a_i is a set variable, and is '$\langle a_i \rangle$' in case a_i is a class variable, then the consistency of K with axioms A, B, C cannot be proved by the construction of a particular normal complete model, if K is a consequence of A, B, C, D, '$V = L$', '$\sim(\exists c, s)\mathrm{PCM}(c, s)$'. Hence it is impossible to show the independence of axiom D, '$V = L$',

'$\sim(\exists c, s)\mathrm{PCM}(c, s)$', '$\mathrm{In}_1=0$', '$\mathrm{In}_2=0$', the generalized continuum hypothesis or any statement which is a consequence of these (such as the axiom of choice), by the construction of a particular normal complete model. Moreover, if A, B, C are ω-consistent, these results follow for all complete inner models, and therefore for all models which satisfy i), ii), iii), iv). Lastly, we have the following theorem: If K is a proposition of the form '$(Qa_1)\cdots(Qa_r)\mathfrak{A}$' where '$\mathfrak{A}$' is a quantifier-free formula built up from contexts of the form '$\mathfrak{M}(a_i)$', '$a_i \varepsilon a_j$', and $a_1, \cdots, a_r$ are all the variables occurring in '$\mathfrak{A}$' and are all set variables and '(Qa_i)' is either '(a_i)' or '$(\exists a_i)$', then if a normal complete model can be constructed and proved on the basis of A, B, C, D, '$V=L$', '$\sim(\exists c, s)\mathrm{PCM}(c, s)$' to satisfy axioms A, B, C, and also K, then this furnishes a proof that K is not merely consistent with A, B, C, D, '$V=L$', '$\sim(\exists c, s)\mathrm{PCM}(c, s)$', but is in fact deducible from them. If A, B, C are ω-consistent, then this holds for all complete models, and hence for all models which satisfy i)–iv). *I. Novak Gál* (Ithaca, N. Y.).

Dresden, Arnold. Complete independence. Scripta Math. 19, 205–206 (1953).

If P and Q are properties such that $(\neg P)\&Q$, $(\neg P)\&(\neg Q)$ and P are all consistent, and $P\rightarrow Q$ is an identity then (trivially) $\neg P$ and Q are independent, but not completely independent in the sense of E. H. Moore [New Haven Math. Colloq., Yale Univ. Press, 1910, pp. 1–150]. *G. Kreisel* (Reading).

Bouligand, G. Sur les principes des mathématiques. Bull. Soc. Math. Belgique **1952**, 18–25 (1953).

Erim, Kerim. The foundations of mathematics. Pakistan J. Sci. 4, 139–143 (1952).

Rabbeno, Giorgio. Elementi di epistemologia matematica e fisica. Ann. Triestini. Sez. 2. (4) 6-7(22-23), 5–41 (1953).

Author discusses the conflicts between intuition and logic, such as arise from the logical concepts of number, infinitesimal, mass, and the constructs of quantum physics. *C. C. Torrance* (Monterey, Calif.).

van der Waerden, B. L. Einfall und Überlegung in der Mathematik. I, II. Elemente der Math. 8, 121–129 (1953); 9, 1–9 (1954).

Antrittsrede, gehalten am 2. Februar 1952 an der Universität Zürich.

ALGEBRA

Flanders, Harley. A note on the Sylvester-Franke theorem. Amer. Math. Monthly 60, 543–545 (1953).

The author gives new proofs of results in a recent note by L. Tornheim [Amer. Math. Monthly 59, 389–391 (1952); these Rev. 14, 939]. The principal result is a new proof of the Sylvester-Franke theorem: the determinant of the pth compound of an $n\times n$ matrix is equal to the determinant, raised to the power $C_{n-1,p-1}$, of the matrix. Proofs employ methods of the Grassmann exterior algebra [see N. Bourbaki, Algèbre multilinéaire, Hermann, Paris, 1948; these Rev. 10, 231]. *G. B. Price* (Lawrence, Kansas).

Fréchet, M. Rectification. Rend. Circ. Mat. Palermo (2) 2, 71–72 (1953).

Cf. same Rend. (2) 1, 11–27 (1952); the error in question is also pointed out in the review in these Rev. 14, 237.

Lowe, R. D., and Zelinsky, D. Which Galois fields are pure extensions? Math. Student 21, 37–41 (1953).

The following theorem is proved: $GF[q^r]$ is generated over $GF[q]$ by a root of an irreducible pure equation $x^r-\beta=0$ if and only if $q-1$ is divisible by every prime factor of f and also by 4 if f is. Moreover, β is then a primitive root in $GF[q]$. [In the line preceding the last displayed formula on p. 41 read pe' instead of $p^{e'}$.] *W. Ledermann* (Manchester).

Abstract Algebra

***Dubreil-Jacotin, M. L., Lesieur, L., et Croisot, R. Leçons sur la théorie des treillis des structures algébriques ordonnées et des treillis géométriques.** Gauthier-Villars, Paris, 1953. viii+385 pp.

The first one-third of this book consists of an exposition of the elements of lattice theory: partial order; lattices and semi-lattices (ordinary and complete); homomorphisms; modular, distributive, semi-modular, and complemented lattices; $\cup$-independence and rank. The principal points of this exposition are along usual lines; the details have however been worked out freshly and well. It is more detailed, more leisurely, and less encyclopedic than Birkhoff's book [Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948; these Rev. 10, 673], to which indeed the reader is referred for further results and for literature citations. A little familiarity with the elements of modern algebra is assumed. Extensive attention is given to cases in which only one of the operations $\cap$ and $\cup$ is assumed to exist or to have specified properties.

In the middle third, these basic results are applied to an analysis of ordered algebraic systems. An outstanding role is played by semi-lattice-ordered demi-groups (cancellation not assumed), called "gerbiers". Thus again much attention is given to non-dual assumptions. Chapter I of this part covers general properties of groups and groupoids with various degrees of ordering, including such notions as positive and prime elements. Chapter II considers residuals, various types of multiplicative closure, and Archimedean elements. Then attention is turned to congruences, including existence of a congruence which has a specified class, to ideals of various algebraic systems, and to the relationships between congruences and ideals. Chapters V and VI cover multiplicative decompositions of elements and systems, and some miscellaneous topics such as homomorphisms and Artin equivalences.

The last part begins with general geometric lattices of finite dimension, with postulates based on properties of linear varieties (primarily the "covering law" point-$\cup$-variety covers variety) and the dimension of unions of points. Then the finiteness condition is removed. Parallelism is defined, and cardinal products studied. Postulating further the dual of the covering law, or equivalently modularity, defines projective lattices, with an alternative development from points and lines; reducibility is investigated. Further modification of postulates leads to affine geometry,

with appropriate relationship to projective geometry. Plane affine geometries are the subject of a separate chapter, with emphasis on coordinates, translations, and Desargues' theorem. The final chapter covers the relation of vector spaces to affine and projective geometries of dimension greater than two.

The tenor of the book is scrupulous attention to detail and to generalization. There are numerous examples and counterexamples, and the relationship of various concepts is carefully discussed. Some readers, especially ones with little background, may find that major results do not stand out from the rest of the work, and that the style in which alternative definitions are presented in the first part (e.g., pp. 59–61) causes momentary confusion. But, on the whole, the presentation is both careful and lucid.

P. M. Whitman (Silver Spring, Md.).

Gericke, Helmuth. Über den Begriff der algebraischen Struktur. Arch. Math. **4**, 163–171 (1953).

The author determines what precisely the notions of algebra and subalgebra should signify in order to have the result of G. Birkhoff that a lattice is complete if and only if it is isomorphic to the lattice of all subalgebras of some algebra. He also examines a modification of the notion of a complete lattice by defining D-lattices. He shows that a lattice is a D-lattice if and only if it is isomorphic to the lattice of all D-subalgebras of some algebra. *O. Frink.*

Matsumoto, Kazuo. On a lattice relating to the intuitionistic logic. J. Osaka Inst. Sci. Tech. Part I. **2**, 97–107 (1950).

In the first part of this paper the author gives different axiomatic characterisations of a (absolutely) pseudo-complemented distributive lattice (lattice A), e.g., by the three axioms $0^*=1$, $1^*=0$, and: $x\cap y\subset z$ implies $x\cap z^*\subset y^*$. Then he gives various axiomatizations for a lattice A in which $x^*\cup x^{**}=1$ identically (lattice B). He also gives a system of axioms for a lattice L, which, as it seems to the reviewer, is equivalent to that for a lattice B. The only lattice L of three elements is given by a matrix of the reviewer [S.-B. Preuss. Akad. Wiss. **1930**, 42–56, esp. p. 56], which the author seems to confuse with intuitionistic logic.

A. Heyting (Amsterdam).

Iseki, Kiyoshi. On lattice theory. J. Osaka Inst. Sci. Tech. Part I. **3**, 25–31 (1951).

In this paper some characterizations of particular types of lattice are derived. G. Birkhoff and the reviewer have shown [Trans. Amer. Math. Soc. **64**, 299–316 (1948); these Rev. **10**, 279] that in a distributive lattice, every irreducible filter (dual ideal) is prime. The author shows more generally that a necessary and sufficient condition that a lattice be distributive is that every irreducible filter be prime. He then shows that a distributive lattice with O and I elements is a Boolean algebra if and only if every irreducible filter is maximal. It is further shown that if a complete lattice with the disjunction property of Wallman satisfies a weak form of the infinite distributive law, it is a Boolean algebra; if a stronger infinite distributive law is satisfied, then the lattice is isomorphic to the Boolean algebra of all subsets of a set. Finally, the author extends a result of the reviewer by showing that an atomic lattice with the Wallman disjunction property is isomorphic to the lattice of all its representative sets. *O. Frink* (State College, Pa.).

Schmidt, Jürgen. Über die Minimalbedingung. Arch. Math. **4**, 172–181 (1953).

By the minimal condition the author means the descending chain condition. By an initial set (Anfang) of a partially ordered set he means a subset containing with an element a all elements $b\leqq a$. He derives some simple relations between these notions, such as the fact that the union of any family of initial sets satisfying the minimal condition also satisfies it. He then defines the rather complicated notions of a recursion R and an "Erfüllung" F of a recursion R, and proves the "recursion theorem", to the effect that if a partially ordered set E satisfies the minimal condition, then each recursion R in E has a unique Erfüllung F. However, if E does not satisfy the minimal condition, then it is shown that there exists a recursion R in E which has no Erfüllung.

O. Frink (State College, Pa.).

Jaffard, Paul. Théorie arithmétique des anneaux du type de Dedekind. II. Bull. Soc. Math. France **81**, 41–61 (1953).

Most of this paper is devoted to the theory of upper classes (s-ideals) of a totally ordered abelian group, which is then applied to the ideal theory of a multiplication ring of Dedekind type, extending results obtained for uniform (einartig) such rings in an earlier paper of the same title [same Bull. **80**, 61–100 (1952); these Rev. **14**, 613].

Let G be a totally ordered abelian group, written additively. Let $\{P_\alpha\}$ ($\alpha\in\Pi$) be the set of prime upper classes $\neq G_+$ of G. Π is regarded as totally ordered by saying $\alpha\leqq\beta$ if and only if $P_\alpha\supseteq P_\beta$. For $a\in G$, (a) and $((a))$ are defined to be the upper classes consisting of all x in G such that $x\geqq a$ and such that $x>a$, respectively. Π contains a least element ω defined by $P_\omega=((0))$. For any $\alpha\in\Pi$, let H_α be the isolated (convex) subgroup of G consisting of the elements of G_+ not in P_α, and their negatives. Let $G_\alpha=G/H_\alpha$. The author constructs the (Cauchy) completion $\hat{G}$ of G as follows. If ω has an immediate successor $\omega+1$ in Π, then $H_{\omega+1}$ is archimedean, and so can be regarded as embedded in the additive group R of real numbers. G is an ordinal product of $H_{\omega+1}$ and $G_{\omega+1}$. $\hat{G}$ is defined to be the ordinal product of R and $G_{\omega+1}$ having the same factor-set. If ω has no immediate successor, then G is taken to be the projective limit of the groups $G_\alpha(\alpha\in\Pi)$.

Let A be an upper class of G. There exists a greatest index α in Π such that $x\in A$ if and only if $\varphi_\alpha(x)\in\varphi_\alpha(A)$, where φ_α is the canonical homomorphism of G onto G_α; α is called the characteristic index of A. Moreover, A determines within the completion $\hat{G}_\alpha$ of G_α an element a_α such that $\varphi_\alpha(A)=(a_\alpha)$ if G_α is discrete, and such that $\varphi_\alpha(A)=(a_\alpha)$ or $((a_\alpha))$ otherwise. a_α is called the characteristic element of A. A is called quasi-closed if $\varphi_\alpha(A)=(a_\alpha)$ and quasi-open if $\varphi_\alpha(A)=((a_\alpha))$. Thus any upper class A of G is determined, to within at most two possibilities, by its characteristic index α and element a_α. For any $\beta>\alpha$, $\varphi_\beta(A)=(a_\beta)$ with $a_\beta\in G_\beta$, and a_β is called the characteristic element of A relative to the index β.

Let A and B be upper classes of G, and let α and β be their respective characteristic indices. Then $A+B$ and $A:B$ have characteristic index $\gamma=\sup(\alpha,\beta)$. The characteristic element of $A+B$ is the sum of those of A and B relative to the index γ; that of $A:B$ is the difference thereof, except when $\alpha<\beta$ and G_β is discrete. There exists an upper class X such that $A=B+X$ if and only if (1) $\alpha\geqq\beta$, and (2) if $\alpha=\beta$, and B is quasi-open, then so is A. If the equation $A+B=A+X$ does not admit an infinite number of solu-

tions X, it admits at most two. Necessary and sufficient conditions are given that it admit only $X=B$. The semigroup $\mathfrak{S}$ of upper classes of G is the class sum of groups; in fact, the set of all quasi-closed (quasi-open) upper classes of characteristic index α is a group isomorphic to $G_\alpha(\hat{G}_\alpha)$.

These results are now applied to the ideal theory of a multiplication ring $\mathfrak{o}$ of Dedekind type. Let $\{\mathfrak{p}_\mu\}$ $(\mu \in M)$ be the set of maximal ideals of $\mathfrak{o}$; let v_μ be the valuation of the quotient field K of $\mathfrak{o}$ of which $\mathfrak{o}_{\mathfrak{p}_\mu}$ is the valuation ring, and let G_μ be the value group of v_μ. For an $\mathfrak{o}$-ideal $\mathfrak{a}$ the number $m(\mathfrak{a})$ of indices $\mu \in M$ such that $v_\mu(\mathfrak{a}) \neq G_{\mu+}$ is finite. $\mathfrak{a} \leftrightarrow \{v_\mu(\mathfrak{a})\}_{\mu \in M}$ is an order-preserving isomorphism between the lattice-ordered semigroup of $\mathfrak{o}$-ideals and the direct sum $\sum_{\mu \in M} \mathfrak{S}_\mu$, where $\mathfrak{S}_\mu$ is the (totally) ordered semigroup of upper classes of G_μ. Combining this with the resutls described above, the author obtains a number of theorems, of which we give two. Given two $\mathfrak{o}$-ideals, $\mathfrak{a}$ and $\mathfrak{b}$, there exists an $\mathfrak{o}$-ideal $\mathfrak{x}$ (possibly fractional) such that $\mathfrak{a}=\mathfrak{b}\mathfrak{x}$ if and only if (1) $\mathfrak{a}\mathfrak{a}^{-1} \subseteq \mathfrak{b}\mathfrak{b}^{-1}$, and (2) if, for any $\mu \in M$, the upper classes $v_\mu(\mathfrak{a})$ and $v_\mu(\mathfrak{b})$ of G_μ have the same characteristic index, and if $v_\mu(\mathfrak{b})$ is quasi-open, then so is $v_\mu(\mathfrak{a})$. If the equation $\mathfrak{a}\mathfrak{b}=\mathfrak{a}\mathfrak{x}$ does not admit an infinite number of solutions, then it admits at most $2^{m(\mathfrak{a})}$.

A. H. Clifford.

Blackett, D. W. Simple and semisimple near-rings. Proc. Amer. Math. Soc. **4**, 772–785 (1953).

A set N, satisfying all of the usual ring axioms with the possible exceptions of the right distributive law $((n+n')n''=nn''+n'n'')$ and the commutative law of addition, is called a near-ring. Thus the single-valued transformations of an arbitrary group (written additively) into itself form a near-ring. A near-ring homomorphism of a near-ring N into a near-ring N' is a mapping τ such that $(n+n')\tau=n\tau+n'\tau$ and $(nn')\tau=(n\tau)(n'\tau)$. The two-sided ideals of N are defined as the kernels of near-ring homomorphisms of N, and it turns out that they are just the additive normal subgroups T of N such that $NT \subseteq T$ and $(n+t)n'-nn' \in T$, where $n, n' \in N$ and $t \in T$. The additive subgroups M of N such that $MN \subseteq N$ are called the right modules of N. The right ideals of N are defined as the kernels of the N-homomorphisms of the additive group of N and it turns out that they are just the additive normal subgroups R of N such that $(r+n)n'-nn' \in R$, where $n, n' \in N$, $r \in R$. In an ordinary ring, right ideals and right modules are identical notions. This is in general not so for near-rings. However, the left ideals of N, which are defined similarly, are just the additive normal subgroups L of N such that $NL \subseteq L$. The sets of left, right and two-sided ideals are closed under finite sums. This is not necessarily true for right modules.

The author's aim is to develop a structure theory for semi-simple near-rings (i.e. near-rings without nonzero nilpotent right modules) which would run parallel to the classical Wedderburn theory of semi-simple algebras. He succeeds in doing this to a large extent under the assumption that the right modules satisfy the descending chain condition and that $0n=0$ for all elements n of the near-ring. Thus, e.g., representations of right ideals of N as direct sums of minimal right ideals and the construction of the attached orthogonal idempotents are derived, as well as the representation of N as a direct sum of simple near-rings, etc.

J. Levitzki (Jerusalem).

Schafer, R. D. A remark on finite simple rings. Amer. Math. Monthly **60**, 696–697 (1953).

The reviewer has shown that every separable associative algebra over an infinite field is generated by one or two elements according as it is or is not commutative. In the present note the author shows that the result is true for finite simple rings. *A. A. Albert* (Chicago, Ill.).

Nakayama, Tadasi. On a fundamental lemma on weakly normal rings. Proc. Japan Acad. **29**, 191–193 (1953).

A subring S of a ring R is called weakly normal in R if the ring of all endomorphisms of R which commute with the right multiplications by elements of S is generated by the left multiplications and a family of left R-semilinear transformations. The following theorem (and a more complicated generalization) is proved. Let R be a simple ring satisfying the minimum condition. Let S be a weakly normal subring of R which is semisimple, contains the identity element of R, and satisfies the minimum condition. Then R is semisimple as a module for the ring of endomorphisms generated by the right multiplications with elements of S and the left multiplications with elements of R. The case where S is simple was proved earlier by the author [Ann. of Math. (2) **55**, 538–551 (1952); these Rev. **13**, 903] and was used notably in order to complete the Galois theory of simple rings [Trans. Amer. Math. Soc. **73**, 276–292 (1952); these Rev. **14**, 240]. *G. Hochschild* (Urbana, Ill.).

Szendrei, J. On Schreier extension of rings without zero-divisors. Publ. Math. Debrecen **2**, 276–280 (1952).

Let T be a Schreier extension ring of S by R. In the notation of Rédei [Acta. Sci. Math. Szeged **14**, 252–273 (1952); these Rev. **14**, 614] the ring T consists of all pairs (a, α), $a \in R$, $\alpha \in S$, where $(a, \alpha)+(b, \beta)=(a+b, [a, b]+\alpha+\beta)$, $(a, \alpha)(b, \beta)=(ab, \{a, b\}+ab+a\beta+\alpha\beta)$, and $[a, b]$, $\{a, b\}$, ab, $a\beta$ are functions of two variables with values in S satisfying certain conditions. It is proved that T contains no zero divisors if and only if S contains no zero divisors and $a\beta=\alpha\beta$, $a \neq 0$ implies $\beta=0$. If T has no zero divisors, then T contains a unit element if and only if R contains a unit element e and there exists an element γ in S such that $e\alpha=\alpha-\gamma\alpha$ for every element α of S. *F. Kiokemeister.*

Steinfeld, O. Über die Nullteilerfreiheit von Ringen. Publ. Math. Debrecen **2**, 281–285 (1952).

If in the ring R the ideal $\mathfrak{a}$ is a sub-ring without zero divisors, then the left annihilator ideal $\mathfrak{b}$ of any non-zero element of $\mathfrak{a}$ is both left and right annihilator ideal of $\mathfrak{a}$, $\mathfrak{a} \cap \mathfrak{b}=(0)$, and $R/\mathfrak{b}$ contains no zero divisors. It follows that R contains no zero divisors if and only if R contains such an ideal $\mathfrak{a}$ and $\mathfrak{a}$ contains no element which is a right zero divisor in R. This theorem yields immediately the condition quoted in the preceding review that a Schreier extension ring contain no zero divisors. *F. Kiokemeister.*

Goldhaber, J. K. A note on Lie k system automorphisms. Amer. J. Math. **75**, 859–863 (1953).

The author generalizes a theorem of Jacobson and Rickart to the effect that the only additive automorphisms of a simple Lie ring which preserve triple Lie products are automorphisms or anti-automorphisms by showing that a nonsingular linear map of a simple Lie algebra which preserves k-tuple products is a multiple of an automorphism by a $(k-1)$st root of unity. By using this result and the known form of the Lie automorphisms of central simple associative

algebras he also obtains the form of the Lie k-tuple automorphisms of semi-simple separable algebras.

W. G. Lister (Providence, R. I.).

Drazin, M. P., and Gruenberg, K. W. Commutators in associative rings. Proc. Cambridge Philos. Soc. **49**, 590–594 (1953).

The authors are concerned with the refinement of certain results of Jennings [Duke Math. J. **9**, 341–355 (1942); Bull. Amer. Math. Soc. **53**, 593–597 (1947); these Rev. **4**, 69; **9**, 5] concerning the enveloping associative ring of a nilpotent Lie subring. Let R be a ring, R_1 the ideal generated by commutators in R, $[R]$ the Lie ring obtained from R, and let L be a subring of $[R]$ with enveloping associative ring R. The results are: if L is nilpotent and either (1) R_1 is a nilpotent associative ring or (2) R is a finitely generated algebraic algebra of characteristic zero, then the lower central series (Jennings) of R terminates in 0.

W. G. Lister (Providence, R. I.).

Drazin, M. P. Triangular representations of linear algebras. Proc. Cambridge Philos. Soc. **49**, 595–600 (1953).

Let U be a vector space over an algebraically closed field and R a finitely generated algebraic algebra of linear transformations in U. The object of this paper is to show the existence of something like a composition series with 1-dimensional factors for U relative to R whenever R satisfies a weak commutativity-nilpotency condition. This condition is: if $\{a_i\}$ is a sequence of elements in the ideal generated by commutators in R, then for some n, $a_1a_2\cdots a_{n-1}a_n=0$. The central result can be stated as follows: every invariant subspace of U contains a 1-dimensional invariant subspace.

W. G. Lister (Providence, R. I.).

Dynkin, E. B. Normed Lie algebras and analytic groups. Amer. Math. Soc. Translation no. **97**, 66 pp. (1953).

Translated from Uspehi Matem. Nauk (N.S.) **5**, no. 1(35), 135–186 (1950); these Rev. **11**, 712.

Goldberg, S. I. Extensions of Lie algebras and the third cohomology group. Canadian J. Math. **5**, 470–476 (1953).

Let L be a Lie algebra, and let U and W be L-modules. Regarding U, W and U/W not only as L-modules but simultaneously as abelian Lie algebras, suppose there is given a Lie algebra extension E with kernel U/W and factor algebra L. A cohomology criterion is given for the existence of a Lie algebra extension E^* with kernel U and factor algebra L, such that E^*/W, as an extension of U/W by L, is isomorphic with E. In fact, the given extension E corresponds in the well known fashion to a uniquely determined cohomology class c for L in U/W. The coboundary map of the exact cohomology sequence for the pair (W, U) sends c into a 3-dimensional cohomology class c' for L in W, and it is easily seen that E^* exists if and only if $c'=0$.

Contrary to a statement made by the author, this is not analogous to Teichmueller's theory. *G. Hochschild.*

Mori, Mitsuya. On the three-dimensional cohomology group of Lie algebras. J. Math. Soc. Japan **5**, 171–183 (1953).

This is the precise analogue for Lie algebras of the Eilenberg-MacLane theory of group kernels [Ann. of Math. (2) **48**, 326–341 (1947); these Rev. **9**, 7]. The definitions and the results are like those of the group case, the place of the automorphisms being taken by derivations. In particular, it is shown that, for any Lie algebra L and any L-module Z, there is a canonical isomorphism of the group of the similarity classes of L-kernels with center Z into the 3-dimensional cohomology group $H^3(L, Z)$. [Actually, as is shown in a forthcoming paper by the reviewer, this is an isomorphism onto.] *G. Hochschild* (Urbana, Ill.).

Kochendörffer, Rudolf. Zwei Reduktionssätze zum Einbettungsproblem für abelsche Algebren. Math. Nachr. **10**, 75–84 (1953).

Let $\mathfrak{A}$ be a finite abelian group and $\mathfrak{G}$ be an extension of $\mathfrak{A}$ by a finite group $\mathfrak{g}$. If $\mathfrak{A}=\mathfrak{A}^{(1)}\times\mathfrak{A}^{(2)}\times\cdots\times\mathfrak{A}^{(n)}$ is the decomposition of $\mathfrak{A}$ into its Sylow subgroups, a factor set, of $\mathfrak{g}$ in $\mathfrak{A}$, corresponding to the extension $\mathfrak{G}$ is decomposed into component factor sets in $\mathfrak{A}^{(\nu)}$ ($\nu=1, 2, \cdots, n$), which give rise to extensions $\mathfrak{G}^{(\nu)}$ of $\mathfrak{A}^{(\nu)}$ by $\mathfrak{g}$. Let Ω be a field which is galoisian with Galois group $\mathfrak{g}$ over its subfield Ω_0. It is proved first that if there is for each ν a Galois algebra [Hasse, J. Reine Angew. Math. **187**, 14–43 (1949); these Rev. **11**, 576] with abelian Galois group $\mathfrak{A}^{(\nu)}$ (i.e. an abelian algebra) over Ω which is galoisian over Ω_0 with Galois group $\mathfrak{G}^{(\nu)}$, then there exists an abelian algebra with Galois group $\mathfrak{A}$ over Ω which is galoisian over Ω_0 with Galois group $\mathfrak{G}$. Next, let $\mathfrak{s}$ be a subgroup of $\mathfrak{g}$ whose index in $\mathfrak{g}$ is relatively prime to the order of $\mathfrak{A}$, and $\mathfrak{S}$ be the subgroup of $\mathfrak{G}$ corresponding to $\mathfrak{s}$. Let Σ be the subfield of Ω corresponding to $\mathfrak{s}$. Then, if there is an abelian algebra with Galois group $\mathfrak{A}$ over Ω which is galoisian over Σ with Galois group $\mathfrak{S}$, there is an abelian algebra with Galois group $\mathfrak{A}$ over Ω which is galoisian over Ω_0 with Galois group $\mathfrak{G}$. In proving these, Wolf's [Math. Nachr. **9**, 201–216 (1953); these Rev. **15**, 6] foundation of the theory of Galois algebras is employed, and for the latter theorem a theorem of Gaschütz [J. Reine Angew. Math. **190**, 93–107 (1952); these Rev. **14**, 445] is used, by which one expresses factor sets of $\mathfrak{g}$ in $\mathfrak{A}$ by means of those of $\mathfrak{s}$ in $\mathfrak{A}$. *T. Nakayama* (Nagoya).

Lorenzen, Paul. Über die Komplettierung in der Bewertungstheorie. Math. Z. **59**, 84–87 (1953).

For every commutative field K with valuation Ω there exists an extension $\bar{K}$ whose valuation $\bar{\Omega}$ is a prolongation of Ω, such that for every algebraic extension of $\bar{K}$ there is a unique valuation which is a prolongation of $\bar{\Omega}$. For given K and Ω, the completion K^* of K is a field with this property. The present author points out that by its mode of construction, the completion K^* involves features which are alien to algebra. In particular, K^* is not countable even when K is countable. He shows how to find a countable extension $\bar{K}$ of a countable K, such that $\bar{K}$ satisfies the above mentioned condition. The determination of $\bar{K}$ is achieved by the constructivist methods expounded by the author previously [Math. Z. **54**, 1–24 (1951); these Rev. **13**, 310]. The emphasis on the constructive nature of the procedure is essential since the conditions on $\bar{K}$ are satisfied vacuously by the algebraic closure of K. *A. Robinson* (Toronto, Ont.).

Papy, Georges. Sur l'arithmétique dans les algèbres de Grassmann. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. **26**, no. 8, 108 pp. (1952).

This paper is an excellent exposition of decomposition and divisibility in exterior algebra on an arbitrary A-module, generalizing the classical theorems. Proof is here given of the divisibility theorem announced in Bull. Soc. Math. Belgique **3**, 18–31 (1951); these Rev. **13**, 814. The author also implies that results on bivectors under symplectic

transformations announced without proof earlier [Bull. Soc. Math. Belgique 1, 5–14 (1949); Algèbre et théories des nombres, Colloq. Internat. Centre Nat. Recherche Sci., no. 24, Paris, 1950, pp. 187–191; these Rev. 11, 155; 13, 814] are easily established by the methods developed here. *L. C. Hutchinson* (Boston, Mass.).

Vivier, Marcel. Sur la dérivation totale par rapport à une forme quadratique régulière dans l'algèbre extérieure de degré $2n$. C. R. Acad. Sci. Paris 236, 879–881 (1953).

Vivier, Marcel. Sur la structure des formes à multiplication extérieure. C. R. Acad. Sci. Paris 236, 1725–1727 (1953).

The first of these papers gives a more direct way to find certain results given by Lepage [see Algèbre et théorie des nombres, Colloq. Internat. Centre Nat. Recherche Sci. no. 24, Paris, 1950, pp. 181–186; these Rev. 13, 814 and literature there cited] concerning the decomposition of an exterior form F in powers of a regular quadratic form Δ. Also, a condition is given for the divisibility of F by Δ. The calculus is expedited by use of the "derivative" $\partial a_i/\partial\alpha=f$ between two monomials a_i and α, defined as the f necessary to have $\alpha f=a_i$, if f exists, otherwise to vanish (with a similar definition for a total derivative). The second paper studies the structure of forms whose total derivative with respect to Δ is zero. *L. C. Hutchinson* (Boston, Mass.).

Lopšic, A. M. Some problems of tensor algebra in linear dimensionless spaces. Trudy Sem. Vektor. Tenzor. Analizu 6, 365–419 (1948). (Russian)

This paper develops the algebra of tensors (multilinear algebra, complex field) without introducing the concept of the number of dimensions of the space, and consequently without the use of coordinates or the ordinary index notation.

Conditions of divisibility, simplicity, and complanarity for bilinear skew-symmetrie scalar and vector functions (in index terms, tensors of valence 2 and 3, skew-symmetric in two indices) are derived, including a number of new special results. For example: Given two simple bilinear skew-symmetric vector functions R and S, with R not completely simple (R is simple if there exist a linear vector function A and a scalar vector function ϕ so that $Rxy=Ax\phi y$, completely simple if in addition a constant vector a and a scalar function ψ exist so that $Ax=a\psi x$) a necessary and sufficient condition that they have a common vector function factor is that $\underline{\underline{Sxy}}\underline{\underline{Rzu}}\underline{\underline{Rvw}}=0$, where the two underscorings indicate separate alternations on the corresponding variables. *L. C. Hutchinson* (Boston, Mass.).

Gurevič, G. B. The algebra of trivectors. II. Trudy Sem. Vektor. Tenzor. Analizu 6, 28–124 (1948). (Russian)

This paper [of which part I appeared in Trudy Sem. Vektor. Tenzor. Analizu 2–3, 51–118 (1935)] is a systematic study of the invariants and classification of trivectors, together with the labyrinth of algebraic constructions necessary for their derivation. The author's main contributions have been to give invariants characterizing the possible types of trivectors of rank 7 [the forms themselves were given by Schouten, Rend. Circ. Mat. Palermo 55, 137–156 (1931)] and the 13 types and characteristic invariants for rank 8 [C. R. (Doklady) Acad. Sci. URSS 2, 353–356 (1935)] in the field of complex numbers. The present paper supplies proofs not originally given and considerably simplifies cumbersome proofs in the earlier work of himself and others. A wealth of properties of trivectors and associated invariants and comitants is given. A start is also here made in the study of analogous problems for symmetric trivalent tensors, establishing inequalities between the analogous invariants. *L. C. Hutchinson.*

Gurevič, G. B. On a property of the algebra of an arbitrary linear system of polyvectors or symmetric tensors. Trudy Sem. Vektor. Tenzor. Analizu 9, 223–229 (1952). (Russian)

Gurevič, G. B. On a certain linear equation for a trivector. Trudy Sem. Vektor. Tenzor. Analizu 9, 230–235 (1952). (Russian)

In the first paper, a number of theorems relating certain associated contravariant and covariant alternating (and symmetric) tensors are derived. For example: Given any s-vector v there exist solutions v (unique or infinitely many) to

$$v_{ab\cdots c[h}v_{ij\cdots k]}\delta^{ab\cdots cd}=v_{ij\cdots kh}.$$

In the second paper the theorem just quoted is used in the case of the trivector to find for given v all possible non-zero solutions w to $w_{[ijn}v_{xy[a}v_{bc]s]}=0$. Solutions exist if and only if v is one of the types (0), (3), (51), (63), or (741), as defined in the paper reviewed above. *L. C. Hutchinson.*

Gurevič, G. B. On the inclusion of a linear system of polyvectors or of symmetric tensors in a complete system. Mat. Sbornik N.S. 30(72), 225–232 (1952). (Russian)

An algorithm is constructed which associates with any linear system w of s-vectors a complete system $\mathfrak{w}^s$ such that $\mathfrak{w}\subset\mathfrak{w}^s$, $\mathfrak{A}\subset\mathfrak{M}$, where $\mathfrak{A}$ is the (Lie) algebra of $\mathfrak{w}$ and $\mathfrak{M}$ the principle algebra of $\mathfrak{w}^s$, these terms being as defined in Mat. Sbornik N.S. 27(69), 103–116 (1950); these Rev. 14, 586, and literature there quoted. *L. C. Hutchinson.*

Theory of Groups

Evans, Trevor. On multiplicative systems defined by generators and relations. II. Monogenic loops. Proc. Cambridge Philos. Soc. 49, 579–589 (1953).

The author uses techniques developed by him in a previous paper [same Proc. 47, 637–649 (1951); these Rev. 13, 312] to study properties of multiplicative systems generated by one element (monogenic systems). Using the normal-form theorems of the previous paper the author derives fairly complete information concerning the automorphisms and endomorphisms of free and finitely-related monogenic loops and gives a decision method for determining whether two finitely related monogenic loops are isomorphic. An example is given of a monogenic loop having an infinite irredundant set of relations which is isomorphic to a proper factor loop of itself. The author shows that the set of monogenic loops has the cardinal of the continuum by constructing a monogenic loop with an infinite irredundant set of relations having the property that no two subloops which satisfy subsets of these relations are isomorphic. He proves also that any finitely related monogenic loop which is infinite has a set with cardinal c of homomorphic images, no two of which are isomorphic. In a final section on isotopy and free loops, the author proves that two free loops which are isotopic must be isomorphic, and exhibits a non-

monogenic isotope of the free loop on one generator, this isotope, consequently, not itself a free loop. *G. Bates.*

Higman, Graham, and Neumann, B. H. Groups as groupoids with one law. Publ. Math. Debrecen **2**, 215–221 (1952).

Soit E un ensemble donné. Soit $\cdot$ une opération partout définie sur E, et soit $W_1(x_1, \cdots, x_n)$ un mot construit à l'aide de $x_1, \cdots, x_n$ et $\cdot$. Si $\cdot$ est une opération de groupe satisfaisant à la loi universelle $W_1(x_1, \cdots, x_n) = e$ et si l'on définit sur E une opération $/$ par $x/y = x \cdot y^{-1}$, alors $/$ satisfait à la loi universelle:

$$(3.2) \qquad x/\{[((((x/x)/W_2)/y)/z]/[((x/x)/x)/z]\} = y$$

où W_2 est l'expression de W_1 en fonction de $/$.

Réciproquement, soit $/$ une opération partout définie sur E et soit $W_2(x_1, \cdots, x_n)$ un mot construit à l'aide de $x_1, \cdots, x_n$ et $/$. Si $/$ est une opération satisfaisant à la loi universelle (3.2) et si l'on définit sur E une opération $\cdot$ par $x \cdot y = x/((y/y)/y)$, alors $\cdot$ est une opération de groupe satisfaisant à la loi universelle: $W_1(x_1, \cdots, x_n) = e$ où W_1 est l'expression de W_2 en fonction de $\cdot$. On peut souvent gagner en simplicité pour des W particuliers. Par exemple, si $\cdot$ est une opération de groupe abélien et si $/$ est défini par $x/y = x \cdot y^{-1}$, alors $/$ satisfait à la loi universelle (4.1) $x/((y/z)/(y/x)) = z$. Réciproquement si $/$ est une opération satisfaisant à (4.1) et si l'on définit $\cdot$ par etc.

J. Riguet (Paris).

Evans, Trevor, and Neumann, B. H. On varieties of groupoids and loops. J. London Math. Soc. **28**, 342–350 (1953).

The number of varieties of groupoids [see the paper reviewed above] is shown to be $\mathfrak{c}$, the cardinal number of the continuum. This is accomplished by constructing an infinite irredundant set of groupoid laws. The same result is proved for loops. Finally it is shown that power associativity in groupoids, loops, or quasigroups is not equivalent to any finite set of laws. *D. C. Murdoch.*

Matsushita, Shin-Ichi. Sur la puissance des ordres dans un groupe libre. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 15–16 (1953).

Si $F^{(n)}$ est le groupe libre à n générateurs (n fini $\geqq 2$), l'ensemble des rélations d'ordre total sur $F^{(n)}$ compatibles avec sa structure de groupe à la puissance du continu.

J. Riguet (Paris).

Jaffard, Paul. Contribution à l'étude des groupes ordonnés. J. Math. Pures Appl. (9) **32**, 203–280 (1953).

This is a systematic study of partially ordered abelian groups. The chief results were summarized in five earlier notes by the author [these Rev. **11**, 579, 640; **12**, 480, 475]. The definitions and results given in the reviews of these notes will not be repeated here. The chapter headings are: I. Generalities on ordered groups. II. The theory of filets and its applications. III. Demi-valued fields. Chapter I gives an excellent treatment of order-preserving (croissant) homomorphisms, among other topics of a general nature, including extensions of the order relation. In addition to the results announced in the last of the five notes cited, Chapter III contains a determination of all demi-valuations w of a simple transcendental extension $k(x)$ of a field k such that $w(a) = 0$ for all $a \in k$. Chapter II has been considerably expanded, and in the remainder of this review we shall give some of the many results not previously announced.

Two positive elements a, b of a (partially) ordered (abelian) group G are called disjoint (étrangers) if $\inf(a, b) = 0$, and weakly disjoint (which we shall write $a \perp b$) if $0 \leqq x \leqq a, b$ implies $x = 0$. G is called regular if weakly disjoint elements are disjoint. Every lattice-ordered group is regular, but not conversely. The partially ordered set $\mathfrak{F}$ of filets $\bar{a}, \bar{b}, \cdots$ of a regular group G has the properties: (α) $\mathfrak{F}$ has a least element $\bar{0}$; (β) $\mathfrak{F}$ is an upper semi-lattice; (γ) if $\bar{a} \perp \bar{c}$ and $\bar{b} \perp \bar{c}$ then $\sup(\bar{a}, \bar{b}) \perp \bar{c}$; ($\delta$) if $\bar{a}$ is not $\leqq \bar{b}$, there exists $\bar{c} \in \mathfrak{F}$ such that $\bar{c} < \bar{a}$ and $\bar{c} \perp \bar{b}$. If $\bar{x} \in \mathfrak{F}$, let $\mathfrak{M}(\bar{x})$ denote the set of all minimal filets $\bar{a}$ of G such that $\bar{x} \geqq \bar{a}$. An ordered group G is said to have property M if $\mathfrak{M}(\bar{x})$ is not empty for any $x \in G_+$, and property MF if in addition every $\mathfrak{M}(\bar{x})$ is finite. The following four properties on the set $\mathfrak{F}$ of filets of a regular group G are equivalent: (1) weak ascending chain condition, (2) descending chain condition and $\mathfrak{F}$ relatively complemented, (3) property MF, (4) $\mathfrak{F}$ is isomorphic to the set (ordered by inclusion) of finite subsets of a set.

Lorenzen [Math. Z. **45**, 533–553 (1939); these Rev. **1**, 101] has shown that any lattice-ordered (abelian) group G is order-isomorphic with a subgroup G' of a direct product $\Gamma = \prod_{\iota \in I} G_\iota$ of totally ordered groups G_ι such that (1) G' is a sublattice of Γ, and (2) the projection of G' upon G_ι is all of G_ι, for each $\iota \in I$ (i.e., G is a subdirect product of the G_ι). Such a representation of G is called a realization of G. An index $\alpha \in I$ is called superfluous if the canonical homomorphism of Γ onto $\prod_{\iota \neq \alpha} G_\iota$ induces an order-isomorphism of G'; this is the case if and only if $G' \cap G_\alpha = \{0\}$. If no $\alpha \in I$ is superfluous, the realization is called irreducible. The author shows that a lattice-ordered group admits an irreducible realization if and only if it has property M, and then the realization is essentially unique. Several conditions are given under which an ordered group G can be realized as a group of real-valued functions; e.g., if a "para-archimedean", lattice-ordered group G admits an irreducible realization, then each of the constituent totally ordered groups is archimedean. *A. H. Clifford* (Baltimore, Md.).

Scott, Leland L. Finite metabelian groups and planes of Σ_{14}. Duke Math. J. **20**, 405–415 (1953).

Any finite metabelian group, whose elements except identity are of order p, is the direct product of an abelian group C' and a metabelian group G whose commutator subgroup coincides with its center C. The groups C and G have the orders p^e and p^{e+k}, where $C \cong k(k-1)/2 - d$, $d \geqq 0$. With k generators $U_1, \cdots, U_k$ and the commutator relations $C_{ij} = U_i^{-1} U_j^{-1} U_i U_j$ the master group $\mathfrak{G}$ with $d = 0$ is uniquely defined. Placing d additional conditions on the commutators serves to define G as a quotient group of $\mathfrak{G}$ of index p^d. Since H. R. Brahana has classified the groups for $k \leqq 5$, and he and W. A. Ferguson have treated the cases $k = 6$, $d = 0, 1, 2$, the author considers next the classification of metabelian groups $k = 6$, $c = 12$ ($d = 3$). The method is to map the subgroups $\{C, \prod U_i^{x^i}\}$ of G onto points $(x_1 \cdots x_6)$ of a finite projective five-space S_5, and the subgroups of order p, generated by an element $\prod C_{ij}^{a_{ij}}$, $i < j$, of the center C, onto the point $(a_{12} \cdots a_{56})$ of a finite projective fourteen space Σ_{14}. The commutators of G are represented by "V-points" lying in an 8-dimensional submanifold V defined by quadratic identities in the Pluecker coordinates of the lines of S_5. The S_3's and S_4's of S_5 are respectively in 1-1 correspondence with Σ_8's and Σ_9's of Σ_{14}, and each point of Σ_8 or Σ_9 that is not a V-point is a "Q-point" lying on a set of bisecants of V, and determining thereby a unique S_4. Points of Σ_{14} that are not V- or Q-points are equivalent under the

collineation group of Σ_{14} that leaves V fixed, and are called "P-points." The three distinct types of points correspond to abstractly different groups G for $k=6$, $c=14$, whereas the 14 non-equivalent types of lines given by Brahana and Ferguson correspond to different groups G for $k=6$, $c=13$.

Here the problem for $k=6$, $c=12$ is to classify the non-equivalent types of planes in Σ_{14}. Forty-two different types are obtained that contain at least one V-point and at least one line of one of the seven types of lines that do not contain P-points. A table gives a canonical form for each of these 42 types of planes, and the numbers of points and lines of each kind that it contains. Additional comments call attention to special geometric relations in many of the planes.

J. S. Frame (East Lansing, Mich.).

Taketa, Kiyosi. Über die Struktur der metabelschen Gruppen. II. J. Osaka Inst. Sci. Tech. Part I. **2**, 1–28 (1950).

Taketa, Kiyosi. Über die Struktur der metabelschen Gruppen. III. Tôhoku Math. J. (2) **4**, 10–32 (1952).

These papers are the second and third installments of a series, of which the first appeared in Jap. J. Math. **13**, 129–232 (1937). The author uses the word metabelian (two-fold metabelian) to indicate a group $\mathfrak{G}$ with a maximal abelian subgroup $\mathfrak{A}$ such that $\overline{\mathfrak{G}}=\mathfrak{G}/\mathfrak{A}$ is abelian. Each element of $\overline{\mathfrak{G}}$ induces an automorphism of $\mathfrak{A}$ and $\overline{\mathfrak{G}}$ can be represented as a congruence group of matrices with integer elements. The study of the groups $\overline{\mathfrak{G}}$ that can be associated with a given $\mathfrak{A}$ is translated into the study of abelian groups of matrices with elements in a Galois field and after some normalization this is reduced to the study of maximal abelian p-groups $\mathfrak{P}$ of matrices over a Galois field such that each element is a lower triangular matrix A with 1's on the diagonal. The derived group-matrix P of $\mathfrak{P}$ is obtained from the group-matrix of $\mathfrak{P}$ by replacing all the diagonal elements by 0. The study of $\mathfrak{P}$ is then reduced to finding canonical forms for P under various assumptions. The derived group matrix P is split into subdiagonal submatrices P_{ij} and the cases treated are distinguished in terms of the number and arrangement of independent elements in the first rows of the P_{ii}. The typical theorems have the form: If the first rows of the submatrices P_{ii} $(i=1, 2, \cdots, w)$ satisfy condition $\cdots$ then a canonical form for P is $\cdots$ and the order of R is $\cdots$. The author expects to continue his studies in further papers of the series. *R. M. Thrall.*

Gaschütz, Wolfgang. Über die Φ-Untergruppe endlicher Gruppen. Math. Z. **58**, 160–170 (1953).

The subgroups $\Phi=\Phi(\mathfrak{G})$ and $\mathfrak{S}=\mathfrak{S}(\mathfrak{G})$ of a finite group $\mathfrak{G}$ are defined to be the intersections, respectively, of its maximal subgroups, and of its maximal normal subgroups. In 1885 Frattini had proved that the Φ-subgroup is nilpotent (all Sylow subgroups invariant). Letting $\mathfrak{L}(\mathfrak{G})$ be the maximal nilpotent normal subgroup of $\mathfrak{G}$, the author shows that $\mathfrak{L}(\mathfrak{G})/\Phi(\mathfrak{G})$ is the product of minimal abelian normal subgroups of $\mathfrak{G}/\Phi(\mathfrak{G})$. Also $\Phi(\mathfrak{G})$ contains precisely those elements of $\mathfrak{G}$ that are superfluous in any set of generators of $\mathfrak{G}$. After defining the splitting of $\mathfrak{G}$ over a normal subgroup $\mathfrak{N}$ to mean "$\mathfrak{G}=\mathfrak{N}\overline{\mathfrak{G}}$ with $\overline{\mathfrak{G}}$ a proper subgroup of $\mathfrak{G}$," Theorem 1 states that $\mathfrak{N}\subseteq\Phi(\mathfrak{G})$ if and only if $\mathfrak{G}$ does not split over $\mathfrak{N}$, and Theorem 2 that $\Phi(\mathfrak{G}/\Psi)=\Phi(\mathfrak{G})/\Psi$ for any normal subgroup Ψ of $\mathfrak{G}$ contained in $\Phi(\mathfrak{G})$. Any group $\mathfrak{F}$, such as $\mathfrak{G}/\Phi(\mathfrak{G})$, for which $\Phi(\mathfrak{F})=1$, is called "Φ-free." A number of theorems are proved concerning the Φ-subgroups and other related subgroups, including the fact that every finite abelian group but not every finite nilpotent group can be the Φ-subgroup of a group. The author is interested in the role of the Φ-group in a general construction of finite groups like the one proposed by Fitting. The concept of complete reducibility, with respect to a subgroup of the automorphisms of $\mathfrak{G}$, is used in the main Theorem 17, which gives a construction for all "Φ-free" groups $\mathfrak{F}$ having a given direct product of simple groups as $\mathfrak{S}$-subgroup. *J. S. Frame.*

Baer, Reinhold. Nilpotent characteristic subgroups of finite groups. Amer. J. Math. **75**, 633–664 (1953).

This paper contains a study of certain important nilpotent characteristic subgroups of a finite group G, among which, in particular, Frattini's subgroup $\Phi(G)$ and Fitting's subgroup $F(G)$ are thoroughly investigated. The former is the intersection of all the maximal subgroups of G, and the latter is the product of all the normal nilpotent subgroups of G. These two subgroups are quite closely connected by a theorem of Gaschütz which states that $F[G/\Phi(G)]=F(G)/\Phi(G)$ is an elementary abelian group [see the paper reviewed above]. The author gives a new proof of this theorem and deduces from it many important corollaries, some of which are the following. G is nilpotent if and only if $G/\Phi(G)$ is an elementary abelian group. [This is Wielandt's characterization of nilpotent groups.] Let N be a normal subgroup of G satisfying $N\cap\Phi(G)=1$; then N is nilpotent if and only if N is an elementary abelian group. [An abelian group is called elementary if the orders of the elements of the group are all squarefree numbers.] G is nilpotent if and only if G is solvable and $F(G)$ is weakly hypercentral. Here "N is weakly hypercentral" means that N is a normal nilpotent subgroup of G and, for every normal subgroup M of G which contains N, the totality of elements x in M, such that every prime divisor of $o(x)$ is a factor of the index $(M:N)$, is a subgroup. [$o(x)$ denotes the order of x.] The usefulness of this concept is shown by the following two theorems. Suppose that the weakly hypercentral subgroup M of G is a part of the normal subgroup N of G. Then N is nilpotent if and only if N/M is nilpotent. The normal subgroup N of G is weakly hypercentral if and only if N is a nilpotent group whose primary components are weakly hypercentral.

The author proves a number of new theorems also concerning Frattini's subgroup, some of which are the following. The normal subgroup N of G is the Frattini subgroup of G if and only if N has the following properties. (1) If S is a subgroup of G and $G=NS$, then $S=G$. (2) Every abelian minimal normal subgroup of G/N possesses a complement in G/N. [The subgroup B of a group K is called a complement of the normal subgroup A of K if $K=AB$ and $A\cap B=1$.] $\Phi(G)$ is always contained in the "weak hypercenter" $H_w(G)$ of G, the latter being defined as the intersection of all maximal weakly hypercentral subgroups of G; $\Phi(G)$ is weakly hypercentral.

On the basis of the previous results the author proves the following theorem which admits various applications: If M is a minimal normal subgroup of the group G, if $Z(M)$ is the centralizer of M in G and if $G/Z(M)$ contains a normal subgroup, not 1, whose order is prime to the order of M, then M is abelian and there exists a subgroup S of G such that $G=SZ(M)$ and $M\cap S=1$. As applications the author obtains a great number of criteria for solvability and, in the concluding part of the paper, he also continues his previous investigations of n-solvable and n-nilpotent groups

[Math. Ann. 124, 161–177 (1952); Proc. Amer. Math. Soc. 4, 15–26 (1953); these Rev. 13, 622; 14, 722].
T. Szele (Debrecen).

Trofimov, P. I. On the influence of the number of all classes of noninvariant conjugate subgroups on properties of a finite nonspecial group. Mat. Sbornik N.S. 33(75), 45–72 (1953). (Russian)

Let $\mathfrak{G}$ be a finite group of order g, $\tau(g)$ the number of distinct prime factors of g, and $\rho(\mathfrak{G})$ the number of classes of mutually conjugate subgroups not invariant in $\mathfrak{G}$. Results by Sigley [Ann. of Math. (2) 41, 767–770 (1940); these Rev. 2, 125] are improved. For every natural number k, there exists a finite non-special solvable group $\mathfrak{G}$ with $\rho(\mathfrak{G})=k$. If $\mathfrak{G}$ is non-special, then for $\rho(\mathfrak{G})=1$ or 2, $\tau(g)\leqq\rho(\mathfrak{G})+1$; for $\rho(\mathfrak{G})=3$ or 4, $\tau(g)\leqq\rho(\mathfrak{G})$; for $\rho(\mathfrak{G})\geqq 5$, $\tau(g)\leqq\rho(\mathfrak{G})-2$. If $\mathfrak{G}$ is non-special with $\tau(g)=\rho(\mathfrak{G})+s$, where s is 1, 0, or -1, then $\mathfrak{G}$ is solvable. Every finite group $\mathfrak{G}$ for which $\rho(\mathfrak{G})\leqq 6$ is solvable. The bestness of the last result is illustrated by the observation that if $\mathfrak{A}_5$ is the icosahedral group, then $\rho(\mathfrak{A}_5)=7$.
R. A. Good (College Park, Md.).

Čunihin, S. A. On existence and conjugateness of subgroups of a finite group. Mat. Sbornik N.S. 33(75), 111–132 (1953). (Russian)

This paper presents the detailed discussion for the theory summarized in two papers published earlier by the author [Doklady Akad. Nauk SSSR (N.S.) 83, 663–665; 86, 27–30 (1952); these Rev. 13, 818; 14, 350].
R. A. Good.

Honda, Kin'ya. On commutators in finite groups. Comment. Math. Univ. St. Paul. 2, 9–12 (1953).

The author proves the following theorem: Let χ_i $(i=1, 2, \cdots, r)$ be all different simple characters of a finite group G; then an element a of G is a commutator (of two elements) in G if and only if $\sum_{i=1}^{r}\chi_i(a)\chi_i^{-1}(1)\neq 0$. Making use of this result he also proves the following theorem: If an element a of a finite group can be written in the form of a commutator, then any generating element of the cyclic group generated by a can also be written in such a form.
T. Szele (Debrecen).

Zacher, Giovanni. Caratterizzazione dei gruppi risolubili d'ordine finito complementati. Rend. Sem. Mat. Univ. Padova 22, 113–122 (1953).

The author calls a group G complemented if G contains to every subgroup H at least one other subgroup K such that $H\cap K=\{1\}$ and $\{H, K\}=G$. This is a wider class of groups than that considered under the same name by P. Hall [J. London Math. Soc. 12, 201–204 (1937)] who postulates that $HK=G$ so that H and K have to be permutable. Hall proved, among many other things, that finite complemented groups are not only soluble, but supersoluble, i.e. that they have cyclic chief factors. The example of the alternating group A_5 shows that complemented groups (in the wider sense) need not be soluble. The present author gives some characteristic properties of finite soluble groups that are complemented (in the wider sense). A finite soluble group G is complemented if and only if the subgroup N generated by all normal Abelian subgroups of G ("maximal normal special subgroup") is direct product of minimal normal subgroups of G and has a complement in G which is, in turn, a complemented group. If we define by recurrence the normal chain of a group G: $\{1\}=N_0\subseteq N_1\subseteq N_2\subseteq\cdots\subseteq N_r$, where $N_1=N$ and $N_{i+1}/N_i=N(G/N_i)$, $N(G/N_r)=\{1\}$ then the result can also be put into the following form: A finite group G is a soluble complemented group if and only if $N_r=G$, N_{i+1}/N_i is the union of minimal normal subgroups of G/N_i, and the Frattini subgroup of G/N_i is the trivial group. The proofs use results previously obtained by the author in Rend. Accad. Sci. Fis. Mat. Napoli 19, 200–206 (1953) and Rend. Sem. Mat. Univ. Padova 21, 383–394 (1952) [these Rev. 14, 722]. Misprint: on p. 115 in formula (1) read G/N_1 for N_1/G.
K. A. Hirsch (London).

van Est, W. T. Finite groups with generators A, B, C in the relation $A^a=B^b=C^c=ABC=1$. Nieuw Arch. Wiskunde (3) 1, 16–26 (1953). (Dutch. English summary)

The F-group

$$(A_1, A_2, \cdots, A_n: A_1^{a_1}=A_2^{a_2}=\cdots=A_n^{a_n}=1, A_1A_2\cdots A_n=1),$$

$2\leqq a_1\leqq a_2\leqq\cdots\leqq a_n$, is finite cyclic or trivial if $n<3$. If $n=3$ it is a dihedral group if $a_1=a_2=2$, the tetrahedral group if $a_1=2$, $a_2=3$, $a_3=3$, the hexahedral group if $a_1=2$, $a_2=3$, $a_3=4$, and the icosahedral group if $a_1=2$, $a_2=3$, $a_3=5$. Using a combinatorial model related to the "Gruppenbild" the author shows that the group is infinite in all other cases. This result had previously been obtained by W. Threlfall [Abh. Math.-Phys. Kl. Sächs. Akad. Wiss. 41, no. 6 (1932)], using representations by motions in the euclidean, elliptic and hyperbolic planes.
R. H. Fox.

Neumann, B. H., and Neumann, Hanna. A contribution to the embedding theory of group amalgams. Proc. London Math. Soc. (3) 3, 243–256 (1953).

Let A be an amalgam of groups G [for definitions cf. R. Baer, Amer. J. Math. 71, 706–742 (1949); these Rev. 11, 78; and Hanna Neumann, ibid. 72, 671–685 (1950); these Rev. 12, 478]. Let P^* be the group generated by elements a^* corresponding to the elements a in A, with the defining relations $a^*b^*=c^*$ if $ab=c$ in A. The condition **E**: A is embeddable in a group is equivalent to the two conditions **S**: $G_i\cap G_j=(G_i\varphi^*\cap G_j\varphi^*)\varphi^{*-1}$ for $i\neq j$, and **I**: the restriction of φ^* to each G_i is an isomorphism [cf., e.g., B. H. Neumann, 'Anhang' in A. G. Kurosch, Gruppentheorie, Akademieverlag, Berlin, 1953]. In the present paper, the relations between these properties of an amalgam A, together with **E**$^+$: A is embeddable in an abelian group; **A**: A is abelian; and **C**$_n$: A is the amalgam of n constituent groups, are analyzed. It is known that **C**$_3$ implies **E** and **E**$^+$ implies **E** [O. Schreier, Abh. Math. Sem. Hamburg. Univ. 5, 161–183 (1927), and Hanna Neumann, loc. cit.]. The present authors give examples to show that **E** is not a consequence of any of the following sets of conditions: **C**$_3$ and **I**; **C**$_3$ and **S**; **C**$_4$ and **A** and **I**; **C**$_4$ and **A** and **S**. It is furthermore known that **E**$^+$ is a consequence of **C**$_3$ and **A**, but not of **C**$_3$ and **A** and **E** [Hanna Neumann, J. London Math. Soc. 26, 228–232 (1951); these Rev. 12, 800]. In the present paper it is proved that an amalgam A of abelian groups $G_1, \cdots, G_n$ which is embedded in a group P in such a way that each intersection $G_i\cap G_j\cap\{G_1, \cdots, \hat{G}_i, \cdots, \hat{G}_j, \cdots, G_n\}$ lies in the center of P is embeddable in an abelian group, with the consequence that **C**$_4$ and **A** and **E** implies **E**$^+$.
D. G. Higman (Montreal, Que.).

Higman, Graham, and Neumann, B. H. On two questions of Itô. J. London Math. Soc. 29, 84–88 (1954).

Let R be the additive group of rationals, and let P be the group of rational-valued polynomials over R. For $r, s\in R$, $f, g\in P$, the multiplication of ordered pairs defined by $(r, f(x))(s, g(x))=(r+s, f(x+s)+g(x))$ makes the set G of ordered pairs a group, a semi-direct product of R and P.

The authors show that G is a non-nilpotent, countable group with lower central series ending with the derived group P. G coincides with its Frattini subgroup, so that there exists a non-nilpotent Frattini subgroup. Related examples with similar properties are exhibited. It is also proved that the Frattini subgroup of a free product of non-trivial groups is trival so that free products always have maximal subgroups. *F. Haimo* (St. Louis, Mo.).

Smirnov, D. M. On groups with upper central series. Mat. Sbornik N.S. **33(75)**, 471–484 (1953). (Russian)

Section 1 is devoted to the following theorem: if the upper central series of a group has length ω, then at least one of its factors does not satisfy the minimal condition for servant subgroups. Section 2 provides the proofs for some of the results announced in an earlier paper by the author [Doklady Akad. Nauk SSSR (N.S.) **76**, 643–646 (1951); these Rev. **12**, 587]. *R. A. Good* (College Park, Md.).

Fuchs, L., Kertész, A., and Szele, T. On a special kind of duality in group theory. I. Acta Math. Acad. Sci. Hungar. **4**, 169–178 (1953). (Russian summary)

If G is a group, then $S(G)$ is the set of all essentially different subgroups of G and $F(G)$ is the set of all the essentially different homomorphic images of G [where two groups are essentially the same if they are isomorphic]. The authors determine all countable abelian groups A such that $S(A)=F(A)$; they determine all pairs of countable abelian groups A and B such that $S(A)=F(B)$ and $F(A)=S(B)$; and they determine all pairs of countable abelian groups U, V such that $S(U)=F(V)$. *R. Baer.*

Ellis, David. Cross-associativity and essential similarity. Amer. Math. Monthly **60**, 545–546 (1953).

If $(G, \cdot)$ and $(G, \#)$ are groups, then the following three conditions are equivalent: (i) $a \# (b \cdot c)=(a \# b)\cdot c$ for every a, b, and c of G; (ii) $a \cdot (b \# c)=(a \cdot b) \# c$ for every a, b, and c of G; and (iii) there is a y of G such that $a \cdot c=a \# y \# c$ for every a and c of G. If the conditions hold, the groups are isomorphic. It is conjectured that if $(G, +, \cdot)$ and $(G, +, \#)$ are sfields, then (iii) holds.

H. A. Thurston (Bristol).

Itô, Noboru. On the factorizations of the linear fractional group $LF(2, p^n)$. Acta Sci. Math. Szeged **15**, 79–84 (1953).

The author gives a complete survey over all factorisations of the fractional linear groups $LF(2, p^n)$. Two factorisations $G=HK=H'K'$ are not counted as distinct if H is a conjugate of H' or K a conjugate of K' or both. The work depends on a knowledge of the subgroups of the linear fractional groups $LF(2, p^n)$ [see Dickson, Linear groups, Teubner, Leipzig, 1901, particularly Chap. XII], especially on classical results on maximal subgroups and the absence of subgroups of low index.

When $p^n \equiv 1 \pmod 4$, then the groups are not factorizable except in the cases $LF(2, 5)$, $LF(2, 9)$ and $LF(2, 29)$ which have two, seven, and four distinct factorisations, respectively. When $p^n \equiv 3 \pmod 4$, then the groups are uniquely factorizable, even into the product of two non-normal maximal subgroups with co-prime orders. It follows then from a theorem of Szép [Acta Sci. Math. Szeged **14**, 22 (1951); these Rev. **14**, 13] that the groups are simple, and this is perhaps the easiest known proof for these cases. Exceptional cases are $LF(2, 7)$ with five, $LF(2, 11)$ with six, $LF(2, 19)$ with three, $LF(2, 23)$ also with three factorisations.

For $p=2$ and $n \geqq 2$ there are two factorisations, for $n=1$ only one. Incidentally the author shows that a situation which the reviewer had considered "unlikely" [these Rev. **14**, 14] actually occurs in certain groups $LF(2, p^n)$.

K. A. Hirsch (London).

Hajós, György. On the problem of factorization of cyclic groups. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **3**, 1–6 (1953). (Hungarian)

This is a Hungarian version of the author's paper: Acta Math. Acad. Sci. Hungar. **1**, 189–195 (1950); these Rev. **13**, 623.

Dieudonné, Jean. On the structure of unitary groups. II. Amer. J. Math. **75**, 665–678 (1953).

This paper is a continuation of a paper with the same title [Trans. Amer. Math. Soc. **72**, 367–385 (1952); these Rev. **14**, 134]. Terminology and notations are those in the first paper. The author considers a unitary group $U_n(K, f)$ where K is a sfield of characteristic 2 and f is a non-degenerate hermitian form over the n-dimensional space E over K. The form f is said to be "trace-valued" if every value $f(x, x)$ can be written $\xi+\xi^J$ (J is the given involution in K). Denote by K^r the additive group of r-tuples over K and by S the set of symmetric elements of K. Theorem 1: The group $U_n(K, f)$ has a composition series $U_n \supset \Gamma_0 \supset \Gamma$ such that $U_n(K, f)/\Gamma_0$ is isomorphic to $U_m(K, f_1)$, where $m \leqq n$ and f_1 is a non-degenerate trace-valued hermitian form; Γ_0/Γ is an abelian group isomorphic to the additive group K^{m-q} and Γ is an abelian group isomorphic to the additive group $S^q \times K^{q(q-1)/2}$ ($2q \leqq n-m$).

Next let K be an arbitrary sfield and assume f to be trace-valued if K has characteristic 2. Also assume the existence of vectors $a \in E$ such that $f(a, a) \neq 0$. For such a vector a, a "quasi-symmetry" of hyperplane H orthogonal to a is an element of $U_n(K, f)$ leaving invariant every element of H. Theorem 2: The group $U_n(K, f)$ is generated by quasi-symmetries except when $n=2$, $K=F_4$ and $J \neq 1$.

To the assumptions for Theorem 2, add $J \neq 1$ and denote by φ the natural homomorphism of the multiplication group K^* of non-zero elements of K onto the factor group K^*/C, where C is the commutator subgroup of K^*. Theorem 3: For every element $u \in U_n(K, f)$, the determinant [Dieudonné, Bull. Soc. Math. France **71**, 27–45 (1943); these Rev. **7**, 3] of u has the form $\varphi(\gamma^J \gamma^{-1})$.

Finally, the author points out that Theorem 4 of the first paper (proved there for a sfield K with characteristic $\neq 2$) is also valid for characteristic 2. Also a correction is made to the proof in the author's memoir "Sur les groupes classiques" [Hermann, Paris, 1940; these Rev. **9**, 494], that every noncommutative sfield K, for which every element has degree $\leqq 2$ over the center Z of K, is a reflexive sfield. *C. E. Rickart* (New Haven, Conn.).

Feit, W. The degree formula for the skew-representations of the symmetric group. Proc. Amer. Math. Soc. **4**, 740–744 (1953).

If $(a)=(a_1, a_2, \cdots, a_k)$ and $(b)=(b_1, b_2, \cdots, b_k)$ represent partitions with $a_i \geqq b_i$, consider the diagram of the partition (a), containing a_i nodes in the ith row, and delete from this diagram those nodes which correspond to the partition (b). The resulting diagram is called a skew-diagram and its properties have been discussed by G. de B. Robinson et al. It is known that the skew-diagram corresponds to a reducible representation of a symmetric group corresponding

to $\sum g_{\lambda ab}\chi^{(\lambda)}$, where $g_{\lambda ab}$ is the coefficient of $\{b\}$ in the product of S-functions $\{\lambda\}\{a\}$. The author proves that the degree of this reducible representation is $n!\det(z_{ij})$ where $n=\sum(a_i-b_i)$, $z_{ij}=1/(a_j-b_i-j+i)!$. An application of the result is given relating to the values of characters of the symmetric groups. *D. E. Littlewood* (Bangor).

NUMBER THEORY

*Gloden, A. **Table des solutions de la congruence $X^4+1\equiv 0 \pmod p$ pour $600\,000<p<800\,000$.** Luxembourg, 1952. 22 pp. 120 francs belges (chez l'auteur: Luxembourg, rue Jean Jaurès, 11.)

This table is an extension to the range of p indicated in the title of a table in Mathematica, Timişoara **21**, 45–65 (1945); these Rev. **7**, 145. There are some 3700 values of $p=8x+1$ in the range considered. *D. H. Lehmer.*

Horváth, J. Prime numbers. IV. Revista Mat. Elem. **2**, 54–72 (1953). (Spanish)

For parts I–III see same Revista **1**, 21–33, 70–78 (1952); **2**, 21–37 (1953); these Rev. **14**, 726.

*Ghoshal, S. C. **Solution of Fermat's Last Theorem.** Privately printed, Lucknow, 1953. ii+12 pp.

An error has been made in the proof of Lemma 2. *D. H. Lehmer* (Berkeley, Calif.).

Wright, E. M. A class of representing functions. J. London Math. Soc. **29**, 63–71 (1954).

For all integers $n\geqq 1$ and positive x let $\lambda_n(x)$ denote a set of positive continuous functions such that $\lambda_n(x)>x$ and $\lambda_n(x_1)-\lambda_n(x_2)\geqq k(x_1-x_2)$ for $x_1>x_2$. Define

$$\phi_n(x)=\lambda_n(\phi_{n-1}(x))$$

for $n\geqq 1$, and $\phi_0(x)=x$. Let B denote a set of positive integers $\{b\}$. If for some c and every $m\geqq\phi_{n-1}(c)$ there is a b such that $\lambda_n(m)\leqq b<\lambda_n(m+1)-1$ then the set E of real $\alpha\geqq c$ such that $[\phi_n(\alpha)]$ belongs to B for all integers $n\geqq 0$ is not empty. If further for an infinity of n and every $m\geqq\phi_{n-1}(c)$ there are at least two members b and b' of B such that $\lambda_n(m)\leqq b<b'\leqq\lambda_n(m+1)-1$ then E has the power of the continuum. If on the other hand for an infinity of n and every $m\geqq\phi_{n-1}(c)$ there is at least one integer a not in B such that $\lambda_n(m)\leqq a<\lambda_n(m+1)-1$ then E is nowhere dense. If $\phi_n(x)$ is a convex function for all $n\geqq 1$ and if, for some r, B contains no set of r consecutive positive integers, then E has measure zero.

These results are based on a more general approach which we shall not attempt to formulate here. The writer specializes B to be the set of primes and shows how to choose $\lambda_n(x)$ to obtain the prime-representing function of Mills [Bull. Amer. Math. Soc. **53**, 604 (1947); these Rev. **8**, 567] and the extensions of Mills' work by Kuipers [Nederl. Akad. Wetensch., Proc. **53**, 309–310=Indagationes Math. **12**, 57–58 (1950); these Rev. **11**, 644] and Niven [Proc. Amer. Math. Soc. **2**, 753–755 (1951); these Rev. **13**, 321]. Another special case is the prime-representing function of the author [Amer. Math. Monthly **58**, 616–618 (1951); these Rev. **13**, 321]. In all these cases the set E of suitable α is nowhere dense, has the power of the continuum, and has measure zero. *I. Niven* (Eugene, Ore.).

Duparc, H. J. A., and Peremans, W. On certain representations of positive integers. Nieuw Arch. Wiskunde (3) **1**, 92–98 (1953).

Let u and v be positive integers, with $(u,v)=1$. A non-negative integer n is said to be representable if $n=ux+vy$, where x and y are non-negative integers. Let Q^k denote the sum of the kth powers of the integers which are not representable. The authors derive a formula for Q^k in terms of Bernoulli numbers, and give a short proof of some known results. *N. J. Fine* (Princeton, N. J.).

Roth, K. F. On certain sets of integers. II. J. London Math. Soc. **29**, 20–26 (1954).

[For part I see same J. **28**, 104–109 (1953); these Rev. **14**, 536.] Let $(a_{\mu\nu})$ be an $l\times n$ matrix whose elements are integers. The author calls a set of positive integers U an A-set if the l equations $\sum_{\nu=1}^{n}a_{\mu\nu}x_\nu=0$, $\mu=1,2,\cdots,l$, are never satisfied simultaneously by distinct integers $x_1, x_2, \cdots, x_n$ of the set U. Denote by $A(x)$ the greatest number of integers that can be selected from $1,2,\cdots,x$ to form an A-set. The author proves the following theorem. Assume that $\sum_{\nu=1}^{n}a_{\mu\nu}=0$ for $\mu=1,\cdots,l$ and further that among the columns of the matrix there exists l linearly independent ones so that if any one of these is excluded, the remaining $n-1$ columns of the matrix can be divided into two sets so that among the columns of each set there are l linearly independent columns. Then $A(x)=\sigma(x)$. In a previous paper [C. R. Acad. Sci. Paris **234**, 388–390 (1952); these Rev. **13**, 724] the author proved that the matrix $(1,-2,1)$ satisfies $A(x)=\sigma(x)$, i.e., if the set U contains for large x more than cx elements the equation $x_1+x_3=2x_2$, $x_i\in U$, is solvable. Unfortunately, the author's condition does not apply for the matrix $\begin{pmatrix}1&-2&1&0\\0&1&-2&1\end{pmatrix}$, i.e., the author cannot prove that if U contains more than cx elements there are $x_i\in U$, $i=1,2,3,4$, which form an arithmetic progression. The proof uses exponential sums and is similar to the author's previous paper. *P. Erdős* (South Bend, Ind.).

Sandham, H. F. A square as the sum of 7 squares. Quart. J. Math., Oxford Ser. (2) **4**, 230–236 (1953).

Let $r_a(n)$ be the number of representations of n as a sum of a squares, $(n|4)=(-1)^{(n-1)/4}$ for n odd, $=0$ for n even. A. Hurwitz [Math. Werke, Bd. II, Birkhäuser, Basel, 1933, p. 751] has given formulas for $r_3(m^2)$, $r_5(m^2)$ in terms of sums involving σ_1, σ_3 respectively. The author proves

$$r_7(m^2)=14\sum_{r\,\text{odd}}\left\{\sigma_3\left(\frac{m}{r}\right)+8\sigma_3\left(\frac{m}{2r}\right)\right\}r^2(r|4)\mu(r),$$

where $\sigma_3(x)=0$ if $x\neq$ integer. The proof is based on the following elementary theorem: If $N(pn)=N(p)N(n)-f(p)N(n/p)$ for p prime, with $f(rs)=f(r)f(s)$, and if $N(0)=0$ or $N(p)=1+f(p)$, then the coefficient of q^{m^2} in

$$(1+2q^{1^2}+2q^{2^2}+2q^{3^2}+\cdots)(N(0)+N(1)q+N(2)q^2+\cdots)$$

is equal to $\sum_r M(2m/r)f(r)\mu(r)$, where

$$\{\sum_{k\geqq 0}N(k)q^{k^2}\}^2=\sum_{n\geqq 0}M(n)q^n.$$

This theorem is applied to identities derived from elliptic functions. [For $N(x)$, $M(x)$, the convention is the same as for $\sigma_3(x)$.] *N. J. Fine* (Princeton, N. J.).

Sandham, H. F. A square as the sum of 9, 11 and 13 squares. J. London Math. Soc. **29**, 31–38 (1954).

[For definitions, see the preceding review.] The author gives formulas for $r_9(m^2)$, $r_{11}(m^2)$, $r_{13}(m^2)$ in terms of divisor sums and arithmetical functions of the type studied by Glaisher and Ramanujan. The method is similar to that used in the preceding paper, but uses an extension of the elementary theorem given there. *N. J. Fine.*

Prachar, Karl. Über Zahlen der Form a^2+b^2 in einer arithmetischen Progression. Math. Nachr. **10**, 51–54 (1953).

The author considers the problem of determining the number of elements $A(x)$ of the arithmetic progression $1+Dn$, $n=1, 2, \cdots$, which are less than x and representable as a sum of two squares. Following the classical procedure of Landau [Handbuch der Lehre der Verteilung der Primzahlen, Bd. 2, Teubner, Leipzig-Berlin, 1909, Kap. 55–58], he considers the associated Dirichlet series. The asymptotic relation is, as in Landau's case, $A(x)\sim bx/\log x$. There is now a certain amount of difficulty in determining the constant b. *R. Bellman* (Santa Monica, Calif.).

Richert, H.-E. On the difference between consecutive squarefree numbers. J. London Math. Soc. **29**, 16–20 (1954).

The following theorem is proved. Let $s_n(q, l)$ denote the nth squarefree number in the arithmetic progression $mq+l$, $1\leqq l\leqq q$, $(q, l)=1$, $m=0, 1, \cdots$, $q=O(n^{2/7})$. Then as $n\to\infty$, $s_{n+1}(q, l)-s_n(q, l)=O(q^{11/9}n^{2/9}\log n)$, uniformly in n, q, and l. In particular, if s_n is the nth squarefree number, then $s_{n+1}-s_n=O(n^{2/9}\log n)$. *L. Carlitz.*

Mahler, K. On the greatest prime factor of ax^m+by^n. Nieuw Arch. Wiskunde (3) **1**, 113–122 (1953).

In this note the following result is established: Let $m\geqq 2$, $n\geqq 3$, $a\neq 0$, $b\neq 0$ be four integers, and let x and y be two integral variables which are relatively prime. Then, as $\max(|x|, |y|)$ increases indefinitely, the greatest prime factor of ax^m+by^n tends to infinity. The proof follows the idea which Landau has applied for the proof of Theorem 695 in his Vorlesungen über Zahlentheorie [Bd. 3, Hirzel, Leipzig, 1927, pp. 60–64]; however, instead of the Thue-Siegel theorem, Mahler uses its $\mathfrak{p}$-adic generalization. In the proof the condition "$(x, y)=1$" is replaced by "(x, y) is bounded", and it is shown that still less is required.
J. F. Koksma (Amsterdam).

Tietze, Heinrich. Über die Glaisher'sche Verallgemeinerung eines Euler'schen Satzes über Partitionen. J. Reine Angew. Math. **191**, 64–68 (1953).

In a previous note [S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1947**, 45–46; these Rev. **9**, 571] the author has given a proof (not new) of Glaisher's generalization of Euler's theorem on partitions into odd parts and distinct parts. In this paper he gives an earlier proof, which does not use generating functions, together with a generalization of Glaisher's result. There are some remarks about "simplicity" of proofs. *N. J. Fine* (Princeton, N. J.).

Rieger, G. J. Zur Hilbertschen Lösung des Waringschen Problems: Abschätzung von $g(n)$. Arch. Math. **4**, 275–281 (1953).

The author shows how Hilbert's method of proof of Waring's conjecture can be used to obtain an upper bound for $g(n)$, the smallest number for which all integers are representable by $g(n)$ nth powers. *W. H. Simons.*

Jamieson, A. M. An expression for Bernoulli numbers. Proc. Glasgow Math. Assoc. **1**, 126–128 (1953).

The author derives an expression for the nth non-zero Bernoulli number as a determinant of the nth order. This is done by expressing the series $\sum_{r=1}^{\infty}(2r-1)^{-2n}\cos(2r-1)x$ as a polynomial in x and "solving" a system of linear equations. Reference is made to Muir's "Theory of determinants" [v. 3, Macmillan, London, 1920, pp. 232–237] for similar results. *D. H. Lehmer* (Los Angeles, Calif.).

Knödel, Walter. Eine obere Schranke für die Anzahl der Carmichaelschen Zahlen kleiner als x. Arch. Math. **4**, 282–284 (1953).

A Carmichael number is a composite integer c which divides $a^{c-1}-1$ for every a prime to c. In a previous paper [Math. Nachr. **9**, 343–350 (1953); these Rev. **14**, 1062] the author showed that the number of such numbers $c<x$ is $O(x\cdot 2^{-t})$ where $t=(\log x)^{1/2}$. Using a theorem of de Bruijn [Nederl. Akad. Wetensch. Proc. Ser. A. **54**, 50–60 (1951); these Rev. **13**, 724] on the number of numbers $<x$ which are free of prime factors $>y$ he improves the above result to read

$$o(x\exp\{-k(\log x\log\log x)^{1/2}\})$$

where $k=2^{-1/2}-\epsilon$, $\epsilon>0$. Use is made of the fact that all Carmichael numbers divisible by a prime q belong to a single residue class modulo $q(q-1)$. *D. H. Lehmer.*

Bailey, W. N. An expression for $\vartheta_1(nz)/\vartheta_1(z)$. Proc. Amer. Math. Soc. **4**, 569–572 (1953).

The author derives the formula

$$\frac{\vartheta_1\{(n-1)z\}}{\vartheta_1(z)}=\prod_{s=1}^{n-1}\frac{\vartheta_1(nz+s\tau\,|\,n\tau)}{\vartheta_1(s\tau\,|\,n\tau)}\cdot\sum_{r=1}^{n-1}\frac{\vartheta_1(r\tau\,|\,n\tau)}{\vartheta_1(nz+r\tau\,|\,n\tau)}e^{(n^2-n-2r)\pi i z},$$

which for $n=3$ reduces to a previous result of his [Proc. London Math. Soc. (3) **1**, 217–221 (1951); these Rev. **13**, 327]. *N. J. Fine* (Princeton, N. J.).

Carlitz, L. Applications of some basic identities. Quart. J. Math., Oxford Ser. (2) **4**, 173–177 (1953).

As is well known, Gauss [Werke, vol. 2, Ges. Wiss. Göttingen, 1876, pp. 11–45] employed a polynomial identity to deduce the formula

$$\sum_{s=0}^{n-1}\epsilon^{s^2}=\prod_{r=1}^{\frac{1}{2}(n-1)}(\epsilon^{2r-1}-\epsilon^{-(2r-1)}),$$

where n is odd and ϵ denotes a primitive nth root of unity. The present author derives similar formulas from known basic identities. A typical result is the formula

$$\prod_{r=1}^{m-1}(1+\alpha^r x)=\sum_{r=0}^{m-1}\frac{(1+\alpha)\cdots(1+\alpha^r)}{(1-\alpha)\cdots(1-\alpha^r)}x^r,$$

where α is a primitive $2m$th root of unity.
A. L. Whiteman (Princeton, N. J.).

Čudakov, N. G. On a class of completely multiplicative functions. Uspehi Matem. Nauk (N.S.) **8**, no. 3(55), 149–150 (1953). (Russian)

Suppose t is a non-zero real number and χ is a non-principal residue-character, and let h be the function defined on the positive integers by the equality $h(n)=\chi(n)n^{it}$. The author proves that h is not a residue-character but nevertheless shares with residue-characters the property that $\sum_{n\leqq x}h(n)$ is bounded. In the terminology of earlier papers [Čudakov and Rodosskiĭ, Doklady Akad. Nauk.

SSSR (N.S.) **73**, 1137–1139 (1950); these Rev. **12**, 393; Čudakov and Linnik, ibid. **74**, 193–196 (1950); these Rev. **12**, 393; Kubilyus and Linnik, Trudy Mat. Inst. Steklov. **38**, 170–172 (1951); these Rev. **15**, 103], this shows that the function h is a generalized character which is not a residue-character and which has an infinite basis. It is unknown whether or not there exist other generalized characters with these two properties. *P. T. Bateman* (Urbana, Ill.).

Koecher, Max. Über Dirichlet-Reihen mit Funktionalgleichung. J. Reine Angew. Math. **192**, 1–23 (1953).

The Dirichlet series under consideration are generalizations of Epstein's zeta function suggested by C. L. Siegel's work on the analytic theory of quadratic forms. If $\mathfrak{S}$ is a positive symmetric $m\times n$ matrix, define

$$(1)\qquad \zeta_n(\mathfrak{S}^{(m)}, s)=\sum_{\mathfrak{A}}|\mathfrak{A}'\mathfrak{S}\mathfrak{A}|^{-s},\quad m\geqq n,$$

where the summation extends over a complete system of non-right-associate integral $m\times n$ matrices $\mathfrak{A}$ of rank n. As usual, $\mathfrak{A}'$ denotes the transpose of $\mathfrak{A}$ and $|\mathfrak{A}|$ the determinant. The author establishes the absolute convergence of (1) for $\mathrm{Re}(s)>m/2$, uses Siegel's generalization of the integral representation for the gamma function to obtain the analytic continuation of (1) in the whole s-plane and then derives the functional equation

$$R_n(\mathfrak{S}, s)=|\mathfrak{S}|^{-s+\frac{1}{2}m}R_n(\mathfrak{S}^{-1}, \tfrac{1}{2}m-s),$$

where

$$(2)\quad R_n(\mathfrak{S}, s)=\pi^{n(n-1-4s)/4}\Gamma(s)\Gamma(s-\tfrac{1}{2})\cdots \times\Gamma(s-\tfrac{1}{2}(n-1))\zeta_n(\mathfrak{S}, s).$$

The difficulties which distinguish the previously known case $n=1$ are side-stepped here by using induction on n. When $m=n$, the series in (1) reduces to a product of Riemann zeta functions:

$$\zeta_n(\mathfrak{S}^{(n)}, s)=|\mathfrak{S}|^{-s}\zeta(2s)\zeta(2s-1)\cdots\zeta(2s-n+1).$$

Taking $\mathfrak{S}$ to be the unit matrix in this case leads to a particularly simple derivation of Minkowski's formula for the volume of the domain of reduced positive quadratic forms with determinant $\leqq 1$.

The methods used here also suffice to extend Hecke's theory (on the connection between modular forms and Dirichlet series with certain types of functional equations) to the modular forms of nth degree considered by Siegel [Math. Z. **43**, 682–708; **44**, 398–426 (1938)]. The Fourier development of a modular form f of degree n and weight k is obtained from a series of the form $\sum A(\mathfrak{T})\exp(2\pi i\,\mathrm{trace}\,\mathfrak{T}\mathfrak{Z})$, $\mathfrak{Z}=\mathfrak{Z}'=\mathfrak{X}+i\mathfrak{Y}$, $\mathfrak{Y}>0$, by putting $\mathfrak{Z}=i\mathfrak{Y}$. The summation extends over all half-integral non-negative symmetric $n\times n$ matrices. The associated Dirichlet series is

$$D_n(f; s)=2^{-ns}\sum a(\mathfrak{T})|\mathfrak{T}|^{-s},\quad a(\mathfrak{T})=A(\mathfrak{T})/E(\mathfrak{T}),$$

where $E(\mathfrak{T})$ denotes the number of units of $\mathfrak{T}$ and the summation now extends over a complete system of non-equivalent positive matrices. If one now forms $R_n(f; s)$ according to (2) with $\zeta_n(\mathfrak{S}, s)$ replaced by $D_n(f; s)$, the functional equation $R_n(f; s)=(-1)^{nk/2}R_n(f; k-s)$ holds. Using different methods, Maass [Math. Ann. **122**, 90–108 (1950); these Rev. **12**, 319] has recently considered the special case $n=2$. The converse problem of characterizing a modular form of degree n by the associated Dirichlet series is also treated. In contrast to Hecke's theory, there is no unique correspondence in the general case. *T. M. Apostol.*

Selmer, Ernst S. Sufficient congruence conditions for the existence of rational points on certain cubic surfaces. Math. Scand. **1**, 113–119 (1953).

It is known that, if a ternary non-homogeneous cubic equation with rational coefficients cannot be homographically reduced to one involving less than three variables, and if it has one rational solution, then the equation has an infinity of rational solutions [B. Segre, J. London Math. Soc. **18**, 226–233 (1943); these Rev. **6**, 37]. Necessary and sufficient conditions for the existence of one solution are not known; but L. J. Mordell [Publ. Math. Debrecen **1**, 1–6 (1949); these Rev. **11**, 82] has conjectured that the required conditions coincide with the elementary congruence conditions.

The present paper proves Mordell's conjecture for the (homogeneous quaternary) equation

$$(1)\qquad a_1x_1^3+a_2x_2^3+a_3x_3^3+a_4x_4^3=0\quad (a_1a_2a_3a_4\neq 0),$$

under the assumption that (for instance) $(a_3a_4)/(a_1a_2)$ is a rational cube. Then the equation (1) can be transformed into $x^3+my^3=n(u^3+mv^3)$, with integer cubefree m and n. The solubility of this equation when the congruence conditions are satisfied is first obtained by means of elementary methods of ideal theory, involving class-number considerations, in the hypothesis that a not-too-high power of 3 divides the class number. Then an alternative proof, unrestrictedly valid, is given by using a deep result due to Hasse [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1931**, 64–69] concerning norms in a cyclic field. The same method can also be applied for proving Mordell's conjecture for the cubic equation $f(x, y)=nf(u, v)$—where f is an arbitrary cubic form with rational coefficients, and so this equation includes the last one—but it fails for the more general equation (1). However, the author affirms to have verified the solubility of equation (1) in all cases, satisfying the congruence conditions, for which $|a_1a_2a_3a_4|\leqq 500$, and he points out some considerations suggesting the same result as highly probable also in remaining cases. *B. Segre.*

Rosen, S. Modular transformation of certain series. Duke Math. J. **20**, 593–599 (1953).

If $\eta(\tau)=x^{1/24}\prod_{n=1}^{\infty}(1-x^n)$, $x=e^{2\pi i\tau}$, denotes Dedekind's modular function, then $[\eta(2\tau)/\eta(\tau)]^{24}$ is invariant under the sub-group consisting of those transformations $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ of the modular group in which c is even. This invariance gives rise to a transformation formula for the function $f(x)=\prod_{n=1}^{\infty}(1+x^n)=e^{-\pi i\tau/12}\eta(2\tau)/\eta(\tau)$, this being the generating function for the number Q_n of partitions of n into unequal parts. Hua [Trans. Amer. Math. Soc. **51**, 194–201 (1942); these Rev. **3**, 270] has used this functional equation to obtain a convergent series for Q_n. Hua also showed that if Q_n is defined by its series expansion, then the function $1+\sum Q_nx^n$ is the same as $f(x)$. The author derives the functional equation for $f(x)$ directly from Hua's series for Q_n by extending a method developed by Rademacher [Amer. J. Math. **61**, 237–248 (1939); **64**, 456 (1942); these Rev. **3**, 271] in studying a similar problem concerning $J(\tau)$. The problem is more involved in the case of $f(x)$, partly because of the presence of certain complicated roots of unity in the transformation equation. *T. M. Apostol.*

Petersson, Hans. Über einen einfachen Typus von Untergruppen der Modulgruppe. Arch. Math. **4**, 308–315 (1953).

Associated with the fundamental region of any sub-group Γ' of the modular group Γ are certain invariants: the genus p, the number σ of non-equivalent parabolic fixed-points, and the numbers e_2 and e_3 which count the number of non-equivalent elliptic fixed-points of orders 2 and 3, respectively. These integers are related to the index of Γ' relative to Γ by the formula $\mu=12(p-1)+6\sigma+3e_2+4e_3$. The sub-groups mentioned in the title of this paper are those with $\sigma=1$ and these the author calls cycloidal sub-groups of Γ. The usual congruence sub-groups employed in the theory of elliptic modular functions have $\sigma\geqq 2$ and are therefore not cycloidal. A theorem of Walburga Rohde is quoted to the effect that when the genus $p=0$, then for every integer $\mu\geqq 1$ there exist several pair-wise non-isomorphic cycloidal sub-groups of Γ which have the same index μ. The author exhibits three cycloidal sub-groups of index $\mu=6$ which are pair-wise non-isomorphic and which correspond to the following values of the invariants: $p=e_2=0$, $e_3=3$; $p=e_3=0$, $e_2=4$; $p=1$, $e_2=e_3=0$. The construction of cycloidal sub-groups is discussed and the following theorem is proved: If Z and Z' are cycloidal sub-groups of Γ with relatively prime indices μ and μ', then $Z\cap Z'$ is a cycloidal sub-group of Γ with index $\mu\mu'$. *T. M. Apostol* (Pasadena, Calif.).

Gagliardo, Emilio. Le funzioni simmetriche semplici delle radici n-esime primitive dell'unità. Boll. Un. Mat. Ital. (3) **8**, 269–273 (1953).

Let $S_k(n)$ denote the sum of the kth powers of the primitive nth roots of unity (Ramanujan's sum); the author proves the formula

$$S_k(n)=\mu\left(\frac{n}{(n,k)}\right)\phi(n)/\phi\left(\frac{n}{(n,k)}\right),$$

which is due to Hölder [Prace Mat.-Fiz. **43**, 13–23 (1936)]. [For a recent generalization of the result see Anderson and Apostol, Duke Math. J. **20**, 211–216 (1953); these Rev. **14**, 951.] *L. Carlitz* (Durham, N. C.).

Carlitz, L. Invariantive theory of equations in a finite field. Trans. Amer. Math. Soc. **75**, 405–427 (1953).

Let $GF(q)$ denote a fixed finite field and r a fixed integer $\geqq 1$. The author considers the set of transformations ϕ: $\xi_i=\phi_i(\eta_1,\cdots,\eta_r)$ $(i=1,\cdots,r)$ possessing an inverse; here $\xi_i, \eta_i \in GF(q)$ and the ϕ_i are polynomials with coefficients in $GF(q)$. It is shown that the totality of such transformations form a group isomorphic with the symmetric group on q^r letters. If $f=f(\xi_1,\cdots,\xi_r)$ is an arbitrary polynomial with coefficients in $GF(q)$ and $\phi f=g$, then f and g are said to be equivalent. Two polynomials f and g are defined as equal if $f(\xi_1,\cdots,\xi_r)=g(\xi_1,\cdots,\xi_r)$ for all ξ_i. Thus the totality of polynomials is separated into a number of classes. The determination of this number turns out to be a simple combinatorial problem.

The author discusses invariantive properties of the equation $f(\xi_1,\cdots,\xi_r)=0$, where f is an arbitrary polynomial with coefficients in $GF(q)$. In particular if $N_f(\alpha)$ denotes the number of solutions of $f=\alpha$, then the numbers $N_f(\alpha)$ form a complete set of invariants in the following sense: Two polynomials f and g are equivalent if and only if $N_f(\alpha)=N_g(\alpha)$ for all $\alpha\neq 0$. Moreover, any invariant of f (with rational values, say) can be exhibited as a polynomial in the $N_f(\alpha)$ with rational coefficients. A number of related topics and applications are also discussed. *A. L. Whiteman.*

Kinohara, Akira. On the derivations and the relative differents in algebraic number fields. J. Sci. Hiroshima Univ. Ser. A. **16**, 261–266 (1952).

The author gives a variant of Kawada's proof [Ann. of Math. (2) **54**, 302–314 (1951); these Rev. **13**, 324] for the fact that the different as defined by A. Weil in terms of derivations coincides with Dedekind's. Let k be an algebraic number field of finite degree, K/k a finite algebraic extension. Denote by i, I the rings of integers of k, K, respectively. If R and S are commutative rings with $S\subset R$, and if T is an R-module, let $\mathfrak{D}(R, S, T)$ stand for the R-module of all derivations of R into T which annihilate S. The author shows that if P is a given prime ideal of I one can find an integral element γ for K/k such that $\mathfrak{D}(I, i, I/P^r)$ is isomorphic with $\mathfrak{D}(i[\gamma], i, I/P^r)$, for every positive integer r. This enables him to prove the equivalence of the two definitions of the different somewhat more directly, though not essentially differently, than was done by Kawada. It is not clear to the reviewer how the corollary to Theorem 2 follows from that theorem or its proof, although it is contained among Kawada's results. *G. Hochschild* (Urbana, Ill.).

Kinohara, Akira. On the derivations and the relative differents in commutative fields. J. Sci. Hiroshima Univ. Ser. A. **16**, 441–456 (1953).

This paper is related to M. Moriya's generalization [Math. J. Okayama Univ. **2**, 111–148 (1953); these Rev. **14**, 952] of Kawada's treatment of the different for algebraic number fields in a similar way as the paper reviewed just above is related to Kawada's. In particular, the appropriate generalization (with the same proof) of the theorem described in the preceding review is used in order to simplify Moriya's proof for the equality of the different defined by derivations with the Dedekind different. Furthermore, the two fundamental theorems on the different (the chain theorem for field towers, and the characterization of the prime divisors of the different) are established directly, without detouring through the Dedekind different.

G. Hochschild (Urbana, Ill.).

Oppenheim, A. One-sided inequalities for quadratic forms. I. Ternary forms. Proc. London Math. Soc. (3) **3**, 328–337 (1953).

Let $f(x, y, z)$ be an indefinite ternary quadratic form with real coefficients in integral variables x, y, z with determinant $\Delta\neq 0$. Let P_1, P_2, P denote the lower bounds of the positive values of f, $-f$ and $|f|$ respectively, so that $P=\min(P_1, P_2)$. Assume $\Delta<0$ without loss of generality. Then Davenport has proved $P_1^3\leqq 4|\Delta|$, $4P_2^3\leqq 27|\Delta|$, equality in the respective cases implying equivalence to a positive multiple of x^2+yz, x^2+4yz [same Proc. (2) **51**, 145–160 (1949); these Rev. **10**, 593]. These results already had been asserted partly by the author [J. London Math. Soc. **6**, 222–226 (1931)] and he now proves Davenport's result and moreover that, if the above mentioned forms are excluded, we shall have $P_1^3\leqq(9/4)|\Delta|$, $P_2^3\leqq(343/64)|\Delta|$, equality implying equivalence to a positive multiple of $3x^2+4yz$, x^2+16yz, respectively. *J. F. Koksma* (Amsterdam).

Watson, G. L. On indefinite quadratic forms in five variables. Proc. London Math. Soc. (3) **3**, 170–181 (1953).

Davenport and Heilbronn proved that any indefinite quadratic form of the type $\lambda_1x_1^2+\cdots+\lambda_4x_4^2$ with real coefficients whose ratios are not all rational, takes arbitrary small values for integers $(x_1,\cdots,x_4)$ not all zero [J. London Math. Soc. **21**, 185–193 (1946); these Rev. **8**, 565]. The

author shows that the same result also holds for the form $\lambda_1x_1^2+\cdots+\lambda_5x_5^2+\lambda_6x_6x_7$, which is supposed indefinite with determinant $\neq 0$ such that the ratios of the real coefficients $\lambda_1, \cdots, \lambda_5$ are not all rational. His main theorem is similar to that of Davenport and Heilbronn [loc. cit.].

J. F. Koksma (Amsterdam).

Watson, G. L. On integers n relatively prime to $[\alpha n]$. Canadian J. Math. **5**, 451–455 (1953).

If α is a real number, then $f(x, \alpha)$ denotes the number of integers n satisfying $1 \leqq n \leqq x$, $(n, [\alpha n]) = 1$. The author shows that $\lim x^{-1}f(x, \alpha)$ exists; it equals $q^{-1}\sum_{u=1}^{q}u^{-1}\varphi(u)$, if α is rational with denominator q (φ is Euler's indicator). If α is irrational, then the limit is shown to be $6\pi^{-2}$. For the proof of the latter case convergents to the continuous fraction of α are used. *N. G. de Bruijn* (Amsterdam).

Estermann, Theodor. On the number of primitive lattice points in a parallelogram. Canadian J. Math. **5**, 456–459 (1953).

An alternative proof is given for the formula $f(x, \alpha) \sim 6\pi^{-2}x$ (α irrational, cf. the preceding review). It depends on estimates for the number of lattice points and for the number of primitive lattice points (m, n) in parallelograms $0 < m \leqq u$, $\alpha m - a < n \leqq \alpha m$. The starting point is, that the number of lattice points in this region is $\sim au$ (a fixed, $u \to \infty$). For this formula the author describes a proof due to Hecke (probably unpublished thus far). Hecke noticed that it is sufficient to deal with the case that a is of the form $\alpha m - n$, where m and n are integers > 0, as the numbers of that form are everywhere dense (according to Kronecker's theorem).

N. G. de Bruijn (Amsterdam).

Few, L. The critical determinant of a displaced convex cylinder. J. London Math. Soc. **29**, 26–30 (1954).

Let K be a star-region in the (x, y)-plane about O, and C the three-dimensional region $(x, y) \in K$, $|z| \leqq 1$. Then $\Delta(C) \leqq \Delta(K)$ and $\Delta(K)/\Delta(C)$ can be arbitrarily large, where $\Delta(C)$, $\Delta(K)$ denote the respective lattice constants for lattices with a point at O [Davenport and Rogers, Quart. J. Math., Oxford Ser. (2) **1**, 215–218 (1950); these Rev. **12**, 161]. If K is convex with centre O then $\Delta(C) = \Delta(K)$ [Yeh, J. London Math. Soc. **23**, 188–195 (1948); these Rev. **10**, 285; Chalk and Rogers, ibid. **23**, 178–187 (1948); these Rev. **10**, 284]. In this paper the author generalizes the proof of Chalk and Rogers to show that $\Delta(C) = \Delta(K)$ if K is any centrally symmetric convex region and O is any inner point of K. *W. S. Cassels* (Cambridge, England).

Bambah, R. P. On lattice coverings. Proc. Nat. Inst. Sci. India **19**, 447–459 (1953).

A homogeneous lattice Λ, of determinant $d(\Lambda) \neq 0$, in n-dimensional Euclidean space R_n is called a covering lattice for a set K if every point of R_n belongs to the set $K+\Lambda$, and the covering constant $c(K)$ is the upper bound of $d(\Lambda)$ for all covering lattices. If there exists a lattice Λ for which $d(\Lambda) = c(K)$ then Λ is called a maximal covering lattice for K. The author proves a great many results on coverings, of which the following is a selection. I. A maximal covering lattice exists when K is a closed bounded set containing O in its interior. II. When K is closed and bounded and Λ is a maximal covering lattice for K, there are n independent points on the boundary of K which are just covered. III. If K is unbounded and closed and has Lebesgue measure $V(K)$ (possibly ∞), then $c(K) \leqq V(K)$. IV. If K is unbounded, closed and contains O in its interior, and if $V(K) < \infty$, then there exists a maximal covering lattice such that the set of all points of R_n which do not lie in $K+\Lambda$ has measure zero. V. There exist star-sets K with $V(K) = \infty$ for which (i) $c(K) < \infty$, (ii) $c(K) = \infty$. VI. $\sup\{V(K)/c(K)\} = \infty$, the supremum being taken over all bounded star-sets K. VII. If $K^{(t)}$ is the set of points of an unbounded star-set K which lie within a distance t of the origin, then $\lim_{t\to\infty} c(K^{(t)}) = c(K)$. From this last result three further theorems are deduced which are analogous to certain results of Mahler and of Davenport and Rogers. *R. A. Rankin.*

***Gel'fond, A. O. Transcendentnye i algebraičeskie čisla. [Transcendental and algebraic numbers.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1952. 224 pp. 7.20 rubles.

According to the author, the aims of this book are to show the contemporary state of the theory of transcendental numbers, to exhibit the fundamental methods of this theory, to present the historical course of development of these methods, and to show the connections which exist between this theory and other problems in the theory of numbers.

The first chapter begins with an historical survey, in which results due to Liouville, Thue, Siegel, Schneider, Mahler, Morduhaĭ-Boltovskoĭ, Kuzmin, Dyson and the author are reviewed. The measure of a number ζ of an algebraic field K, of degree σ and having an integral basis $\omega_1, \cdots, \omega_\sigma$, is then defined as $\min\max(|p_1|, \cdots, |q_\sigma|)$, where the minimum is taken over all representations $\zeta = (p_1\omega_1+\cdots+p_\sigma\omega_\sigma)/(q_1\omega_1+\cdots+q_\sigma\omega_\sigma)$ of ζ. In terms of this measure, the following generalization of Dyson's theorem is proved. I. Let α and β be two arbitrary numbers in an algebraic field K_ν of degree ν. Let ζ and ζ_1 be numbers of an algebraic field K of degree s, whose measures with respect to a fixed integral basis of K are q and q_1, respectively. Let θ and θ_1 be two real numbers such that $2 \leqq \theta \leqq \theta_1 \leqq \nu$, $\theta\theta_1 = 2\nu(1+\epsilon)$, where $\epsilon > 0$ is arbitrarily small but fixed. Then if the inequality $|\alpha - \zeta| < q^{-s\theta}$ has a solution ζ with measure $q > q'(K_\nu, K, \alpha, \beta, \epsilon, \delta)$, the inequality $|\beta - \zeta_1| < q_1^{-s\theta_1}$ cannot have solutions with measure q_1 under the condition that

$$\log q_1 \geqq \left[\frac{\theta-1}{2(\sqrt{(1+\epsilon)}-1)}+\delta\right]\log q,$$

where δ is any arbitrarily small positive constant. The proof, which is given in detail and thus becomes available for the first time outside the USSR [see Gel'fond, Uspehi Matem. Nauk (N.S.) **4**, no. 4(32), 19–49 (1949); these Rev. **11**, 231], uses an improvement of the method of Thue. A p-adic analogue, and other variants, of I are proved. I is then used to prove II: Let $\alpha, \zeta_1, \cdots, \zeta_s$ be numbers of an algebraic field K_ν, and suppose that no product of integral powers of $\zeta_1, \cdots, \zeta_s$ is 1. Then the inequality

$$|\alpha - \zeta_1^{x_1}\cdots\zeta_s^{x_s}| < e^{-\epsilon x},$$

where $x = \max(|x_i|)$, and the congruence $\alpha \equiv \zeta_1^{y_1}\cdots\zeta_s^{y_s} \pmod{\mathfrak{P}^m}$, where $m = [\delta y]$, $y = \max(|y_i|)$, can have, for arbitrary $\epsilon > 0$, $\delta > 0$, only finitely many rational integral solutions $x_1, \cdots, x_s, y_1, \cdots, y_s$. Here $\mathfrak{P}$ is a prime ideal of K_ν. The chapter concludes with some variants of II, and applications of I–II, for some of which see Gel'fond, loc. cit.

The second chapter contains an exposition of Siegel's method, in the form given in the latter's monograph [Transcendental numbers, Princeton, 1949; these Rev. **11**, 330]. The first two sections of the third and last chapter are concerned with the Gel'fond-Schneider theorem, and the work of Schneider on elliptic functions, using Gel'fond's method. In the third section, this method is applied to give

an effective transcendence measure for the $\mathfrak{P}$-adic logarithm, as follows. III. Let a and b be numbers of an algebraic number field K, such that $a^{n_1} \neq b^{n_2}$ for any rational integers n_1 and n_2 not both zero, and let $\mathfrak{P}$ be a prime ideal of K with $N(\mathfrak{P}) = p^r$, $p \equiv 0 \pmod{\mathfrak{P}^e}$. If $|a|_{\mathfrak{P}} = |b|_{\mathfrak{P}} = 1$, then the inequalities

$$\left| \log a^{\lambda} - \frac{n_1}{n_2} \log b^{\lambda} \right|_{\mathfrak{P}} < \mathfrak{P}^{-m_0}, \quad \lambda = p^{3re} - p^{(3p-1)e},$$

$$0 < |n_1| + |n_2| \leqq 2N, \quad m_0 = [\log^7 N], \quad (n_1, n_2) = 1,$$

cannot have solutions for $N > N_0$, where N_0 can be explicitly specified as a function of a, b, and $\mathfrak{P}$. This is followed by an outline of the proof of the author's theorem [loc. cit. pp. 32–36, and Doklady Akad. Nauk SSSR (N.S.) **64**, 277–280 (1949); these Rev. **10**, 682], that if $a \neq 0, 1$ is algebraic and α is a cubic irrationality, then a^{α} and a^{α^2} are algebraically independent over the rationals.

The last two sections of the third chapter include a detailed repetition of the author's work [Uspehi Matem. Nauk (N.S.) **4**, no. 5(33), 14–48 (1949); these Rev. **11**, 231; **12**, 1001] on algebraic independence of certain classes of transcendental numbers. An application to the problem of character sums is noted. *W. J. LeVeque.*

Cassels, J. W. S. A new inequality with application to the theory of diophantine approximation. Math. Ann. **126**, 108–118 (1953).

Let $a_1, a_2, \cdots, a_N$ be a set of real numbers. For $0 \leqq \alpha < \beta \leqq 1$ denote by $F(\alpha, \beta)$ the number of solutions of $\alpha \leqq a_n < \beta \pmod 1$. Then

$$D = \max_{(\alpha,\beta)} \left| \overline{\beta - \alpha} - \frac{1}{N} F(\alpha, \beta) \right|$$

is called the discrepancy of the set. Improving results of Vinogradoff [Izvestiya Akad. Nauk SSSR (6) **20**, 585–600 (1926)] and van der Corput and Pisot [cf. Nederl. Akad. Wetensch., Proc. **42**, 476–486, 554–565, 713–722 (1939) = Indagationes Math. **1**, 143–153, 184–195, 260–269 (1939); these Rev. **1**, 66; and also the author's earlier paper, Proc. Cambridge Philos. Soc. **46**, 642–644 (1950); **48**, 368 (1952); these Rev. **13**, 539, 630], the author proves the fundamental inequality $D \leqq AE^{1/2}(1 + |\log E|)$, where A is an absolute constant and where E denotes the discrepancy of the set of the N^2 differences $a_m - a_n$. Several applications.

J. F. Koksma (Amsterdam).

Godwin, H. J. On a theorem of Khintchine. Proc. London Math. Soc. (3) **3**, 211–221 (1953).

Khintchine [Rend. Circ. Mat. Palermo **50**, 170–195 (1926)] proved the existence of a constant k_1 such that, if α denotes any real number, another real number β can be chosen so that $|x(\alpha x - y + \beta)| \geqq k_1$ for all integers $x > 0$, y. Prasad [Proc. London Math. Soc. (2) **53**, 310–330 (1951); these Rev. **13**, 116] gave an estimate for k_1. Using his argument, the author, by means of simple continued fractions and distinguishing several cases, proved that for any such k_1, $k_1 < 0.2114$, whereas $k_1 = 68/483 = 0.14078$ still is admissable. *J. F. Koksma* (Amsterdam).

Szüsz, P. Verschärfung eines Hardy-Littlewoodschen Satzes. Acta Math. Acad. Sci. Hungar. **4**, 115–118 (1953). (Russian summary)

Let ω be irrational, and let $\alpha_1, \cdots, \alpha_k$ be arbitrary numbers with $0 \leqq \alpha_\kappa < 1$. Put $\{x\} = x - [x]$. It is shown that to each $\epsilon > 0$ there corresponds a positive integer $m = m(\epsilon, k, \omega)$ such that for every positive integer ν there is an integer n with $\nu \leqq n \leqq \nu + m$ for which $|\{n^\kappa \omega\} - \alpha_\kappa| < \epsilon$ for $\kappa = 1, 2, \cdots, k$. This strengthens a well-known theorem of Hardy and Littlewood [Acta Math. **37**, 155–191, 193–239 (1914)]. The proof is by induction on k. *W. J. LeVeque.*

LeVeque, W. J. The distribution modulo 1 of trigonometric sequences. Duke Math. J. **20**, 367–374 (1953).

In an earlier paper by the author [Michigan Math. J. **1**, 139–162 (1953); these Rev. **14**, 1067] there is contained the following result: If $X = (x_1, \cdots, x_r)$ is an r-dimensional continuous variable, if R is a set having positive r-dimensional Lebesgue measure, and if $\{f_n(X)\}$ is a sequence of real-valued functions, such that

$$(1) \quad \left| \int_R \exp\left(i(f_n(X) - f_m(X))\right) dX \right| \leqq \frac{C}{\max(1, |n-m|^\epsilon)}$$

where $C > 0$ and $\epsilon > 0$ are constants, then $\{f_n(X)\}$ is uniformly distributed (mod 1) for almost all $X \in R$. Moreover, we have for each fixed integer $\beta \neq 0$

$$\sum_{n=1}^{N} \exp(2\pi i \beta f_n(X)) = o(N^{1-\frac{1}{2}\epsilon+\theta}), \quad \text{if } 0 < \epsilon < 1,$$

$$= o(N^{\frac{1}{2}} \log^{1/2+\theta} N), \quad \text{if } \epsilon = 1,$$

$$= o(N^{\frac{1}{2}} \log^{1/2+\theta} N), \quad \text{if } \epsilon > 1,$$

θ denoting an arbitrary positive constant. In the present paper some cases are considered in which (1) is fulfilled. If one wants to show that the sequence $\{z^n\}$ for complex $z = re^{i\alpha}$ is uniformly distributed (mod 1), one is lead to the question whether for constant values of B and α the sequence $\{Br^n \cos(nx - \alpha)\}$ (for real x) is uniformly distributed (mod 1). Therefore, more generally, the author investigates (1) for $f_n(x) = u_n \phi(v_n x - \alpha)$, R being the segment $0 \leqq x \leqq \omega$, where $\{u_n\}$ and $\{v_n\}$ are sequences of real numbers and where ϕ denotes a periodic function of period ω satisfying some conditions of regularity. Several applications: $\{z^n\}$ is uniformly distributed for almost all complex z with $|z| > 1$; if $\{u_n\}$ is an increasing sequence of positive integers, then $\{u_n \cos(u_n x - \alpha)\}$ is uniformly distributed (mod 1) for almost all x in $(0, 2\pi)$; etc. *J. F. Koksma.*

ANALYSIS

Černikov, S. N. Systems of linear inequalities. Uspehi Matem. Nauk (N.S.) **8**, no. 2(54), 7–73 (1953). (Russian)

Abbreviated table of contents: Dependence. Consistency. The polyhedron of solutions. Minimal faces. Contact with a plane. Unboundedness. Reducibility. Nondegeneracy. Dimensionality.

The algebraic counterpart to the theory of convex polyhedra, minus duality, is redeveloped. Conditions using signs of minors [see Motzkin, Dissertation, Basel, Jerusalem, 1936, pp. 49–50] are given for the occurrences named in the contents. By nondegeneracy of the polyhedron P of solutions the author means that its dimension is that of the whole space; by reducibility, that P is the vector sum of a bounded polyhedron and of a linear variety.

T. S. Motzkin (Los Angeles, Calif.).

Černikov, S. N. Linear inequalities. Doklady Akad. Nauk SSSR (N.S.) **89**, 977–980 (1953). (Russian)

The author extends his results [Mat. Sbornik N.S. **15(57)**, 437–448 (1944); these Rev. **7**, 109] to general systems of linear inequalities with real coefficients, and gives 11 theorems, without proof, on the consistency of such systems. *T. S. Motzkin* (Los Angeles, Calif.).

Biernacki, Mieczysław. Sur une inégalité entre les intégrales due à Tchébyscheff. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **5** (1951), 23–29 (1953). (Polish and Russian summaries)

Chebyshev's inequality,

$$\int_a^b p(x)f(x)g(x)dx\int_a^b p(x)dx \geqq \int_a^b p(x)f(x)dx\int_a^b p(x)g(x)dx,$$

where $f(x)$ and $g(x)$ are integrable and monotonic in the same sense, and $p(x)$ is positive and integrable, on $[a, b]$, is known [Hardy, Littlewood and Pólya, Inequalities, Cambridge, 1934, p. 43] to hold if $f(x)$ and $g(x)$ are "equally increasing" in $[a, b]$; that is, $[f(x)-f(y)][g(x)-g(y)]\geqq 0$. The author has shown [same Ann. Sect. A. **4**, 123–130 (1950); these Rev. **13**, 117] that the inequality persists if $g(x)$ is nondecreasing and $f(x)$ is nondecreasing "on the average"; that is, the function $\int_a^x p(x)f(x)dx/\int_a^x p(x)dx$ is nondecreasing. He now shows that the inequality still holds if $[a, b]$ can be subdivided into a finite number of subintervals in each of which $f(x)$ and $g(x)$ are equally increasing on the average. *E. F. Beckenbach.*

***Apéry, Roger. Une inégalité sur les fonctions de variable réelle.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 3–4. Casa Editrice Perrella, Roma, 1953.

It is shown that if $f(x)$ is measurable, $0\leqq f\leqq M$, $\int_0^{+\infty} f(x)dx=\omega M$, and $g(x)$ is nonnegative and monotonic decreasing, then $\int_0^{+\infty} f(x)g(x)dx\leqq M\int_0^{\omega} g(x)dx$. Necessary and sufficient conditions for the sign of equality to be taken on are given. *E. F. Beckenbach* (Los Angeles, Calif.).

Gonçalves, J. Vicente. Une idée de Cauchy. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **1**, 405–408 (1951).

The author applies the formula of Cauchy,

$$\frac{\psi(x)-\psi(a)}{\phi(x)-\phi(a)}=\frac{\psi'(x_1)}{\phi'(x_1)} \quad (a<x_1<x),$$

to demonstrate the equality of $\partial^2 f/\partial x\partial y$ and $\partial^2 f/\partial y\partial x$, and also to obtain expressions for remainder terms in finite Maclaurin series. Thus it is shown that if

$$S_i(x)=f^{(i)}(0)+xf^{(i+1)}(0)+\cdots+\frac{x^{n-i-1}}{(n-i-1)!}f^{(n-1)}(0),$$

and $X(x)=[f^{(n-1)}(x)-f^{(n-1)}(0)]/x$ $(0<x<x_0)$, then

$$f^{(n-j)}(x_0)-S_{n-j}(x_0)=\frac{x^j}{j!}X(x_{j-1}) \quad (j=1, \cdots, n),$$

with $0<x_{n-1}<\cdots<x_1<x_0$. *E. F. Beckenbach.*

Sharma, A. On the differentiability of the remainder in Mazzoni's formula. Ganita **2**, 65–67 (1951).

The author gives a result for Mazzoni's form of the mean-value theorem, similar to Whitney's theorem [Duke Math. J. **10**, 153–158 (1943); these Rev. **4**, 192] concerning the differentiability of the remainder term in the Taylor formula. Let $1\leqq n\leqq n+p$, let $f(x)$ be of class C^{n+p+2} in an interval about the origin, and let

$$f(x)=f(0)+\cdots+\frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1}+\frac{x^n}{n!}f^{(n)}\left(\frac{x}{n+1}\right)+\frac{x^{n+2}}{(n+2)!}f_n(x),$$

with $f_n(0)=[n/2(n+1)]f^{(n+2)}(0)$. Then $f_n(x)$ is of class C^p, and is of class C^{p+2} for $x\neq 0$. Explicit formulas are given for the $f_n^{(k)}(x)$, $k=0, \cdots, p$, and it is shown that

$$\lim_{x\to 0} xf_n^{(p+1)}(x)=\lim_{x\to 0} x^2 f_n^{(p+2)}(x)=0.$$

A converse result is given. *E. F. Beckenbach.*

***Tschakaloff, Lubomir. Über den Rolleschen Satz angewandt auf lineare Kombinationen endlich vieler Funktionen.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 591–594. Akadémiai Kiadó, Budapest, 1952. (Hungarian and Russian summaries)

Let the functions $f_0(x), \cdots, f_m(x)$ satisfy the hypotheses of Rolle's theorem for the interval $[a, b]$, and consider the family of functions $F(x)=c_0f_0(x)+\cdots+c_mf_m(x)$, with real parameters $c_0, \cdots, c_m$. It is shown that there exists a $\delta>0$, depending on the functions $f_j(x)$ but not on the parameters c_j, such that for each member $F(x)$ of the family the equation $F'(\xi)=0$ has a solution in the interval $(a+\delta, b-\delta)$. An analogous result is given relative to the law of the mean of the differential calculus. *E. F. Beckenbach.*

Jecklin, H. Trigonometrische Mittelwerte. Elemente der Math. **8**, 54–60 (1953).

The author applies known general results concerning quasi-arithmetic means, or means of the form

$$M=f^{-1}[n^{-1}\sum_{i=1}^{n} f(x_i)]$$

where $f(x)$ is a strictly monotone continuous function and $f^{-1}(x)$ is its inverse, to trigonometric means, where $f(x)$ is a trigonometric function of x, $0<x<\pi/2$. Thus it is shown, for example, that both the cosine mean and the tangent mean dominate the arithmetic mean, and the quadratic mean dominates the cosine mean; but there are values $x_1<x_2$, for which approximations are given, such that the cosine mean dominates the tangent mean for $0<x<x_1$ while the tangent mean dominates the cosine mean for $x_1<x<\pi/2$, and the quadratic mean dominates the tangent mean for $0<x<x_2$ while the tangent mean dominates the quadratic mean for $x_2<x<\pi/2$. *E. F. Beckenbach.*

Tietze, Heinrich. Über eine Verallgemeinerung des Gauss'schen arithmetisch-geometrischen Mittels und die zugehörige Folge von Zahlen-n-tupeln. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1952**, 191–195 (1953).

For n positive numbers $x_1^{(0)}\geqq\cdots\geqq x_n^{(0)}$, with $x_1^{(0)}>x_n^{(0)}$, the n elementary symmetric means (of which the first is the arithmetic mean and the last is the geometric mean) yield numbers $x_1^{(1)}>\cdots>x_n^{(1)}$. Continuation of the process leads to a unique limit value which reduces to the Gaussian arithmetic-geometric mean in the case $n=2$. In the other direction, the values $x_j^{(0)}$ can readily be obtained from the values $x_j^{(1)}$ which they had determined. The author asks whether the process can be further continued backward; that is, whether to an arbitrary set of n positive numbers $x_j^{(0)}$, not all equal, there necessarily corresponds a set of

positive numbers $x_j^{(-1)}$ of which the $x_j^{(0)}$ are the elementary symmetric means. He points out that for $n=2$ an affirmative solution is immediate, but gives a counter-example for $n=3$. He concludes with the question as to whether for $n \geqq 3$ there exists any set which can be continued backward indefinitely.

E. F. Beckenbach (Los Angeles, Calif.).

Chuang, Chi-Tai. Un théorème sur les fonctions convexes croissantes. Acad. Sinica Science Record **5**, 1–9 (1952). (Chinese summary)

For use in a study of the behavior of functions holomorphic in the unit circle, to appear in a subsequent paper, the author establishes a lemma concerning the growth of the members of a class of increasing convex functions.

E. F. Beckenbach (Los Angeles, Calif.).

Hirschman, I. I., Jr. A convexity theorem for certain groups of transformations. J. Analyse Math. **2**, 209–218 (1953).

Etude de la transformation $f(x) \to \Lambda(\sigma)f(x)$ définie pour une fonction $f(x) \in L^p(0, 2\pi)$, $c_n = (2\pi)^{-1}\int_0^{2\pi} f(x)e^{-n\lambda x}dx$, par:

$$f(x) \sim \sum_{-\infty}^{+\infty} c_n e^{i\lambda_n x}, \quad \Lambda(\sigma)f(x) \sim \sum_{-\infty}^{+\infty} e^{\lambda_n \sigma} e^{i\lambda_n x}, \quad -\infty < \sigma < +\infty,$$

λ_n étant une suite de constantes réelles ($-\infty < n < +\infty$). Le principal résultat obtenu exprime une propriété de convexité de $\|\Lambda(\sigma)f\|_n$ par rapport aux variables σ et $1/n$ et s'énonce: Si la transformation est telle que pour toute fonction $g(x) \in L^p(0, 2\pi)$ et toute fonction $h(x) \in L^q(0, 2\pi)$ on ait

$$\|\Lambda(i\tau)g\|_p \leqq A(\tau)\|g\|_p, \quad \|\Lambda(i\tau)h\|_q \leqq A(\tau)\|h\|_q,$$

où $|\tau^{-1}|\log\log A(\tau)$ demeure borné quand $|\tau| \to +\infty$, alors on a

$$\|\Lambda(\sigma)f\|_n \leqq B\|f\|_p^{1-\theta}\|\Lambda(\sigma_1)f\|_q^{\theta},$$

où $0<\theta<1$, $1/n = (1-\theta)/p + \theta/q$, $\sigma = \sigma_1\theta$, et où la constante B ne dépend que de la suite λ_n, de σ, p, q, θ mais non de f. La démonstration consiste comme dans le cas du théorème de Riesz-Thorin [cf. A. Thorin, Comm. Sem. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] **9**, 1–58 (1948); ces Rev. **10**, 21] à se ramener à des propriétés de convexité des fonctions holomorphes.

Application est faite de l'énonce précédent à l'étude de la moyenne $M_r(f, \rho) = [(2\pi)^{-1}\int_0^{2\pi} |f(\rho e^{i\phi})|^r d\phi]^{1/r}$ considérée comme fonction des variables $[\log \rho, 1/r]$, à celle de $\|f_\sigma(x)\|_n$, avec $f_\sigma(x) = \sum_{-\infty}^{+\infty} (c_n/(in)^\sigma)e^{inx}$, pour laquelle on donne une propriété de convexité par rapport à $[\sigma, 1/n]$ et à l'étude plus généralement de $\|f\|_r = [\int_S |f(x)|^r d\mu(x)]^{1/r}$.

P. Lelong (Lille).

Predonzan, Arno. Su una formula d'interpolazione per le funzioni razionali. Rend. Sem. Mat. Univ. Padova **22**, 417–425 (1953).

The problem is discussed and solved of determining a rational function $y = f(x)/g(x)$ subject to the condition of assuming given values $y = y_1, y_2, \cdots, y_{2n+1}$ for $2n+1$ given values of $x = x_1, x_2, \cdots, x_{2n+1}$ respectively, where $f(x)$ and $g(x)$ are polynomials in x of degrees $\leqq n$.

B. Segre.

Davis, Philip. On simple quadratures. Proc. Amer. Math. Soc. **4**, 127–136 (1953).

Une classe de fonctions $f(x)$, intégrables au sens de Riemann dans $0 \leqq x \leqq 1$, possède une formule de quadrature simple lorsqu'il existe deux suites de nombres a_i et λ_i telles que

$$\int_0^1 f(x)dx = \sum_{i=0}^{\infty} a_i f(\lambda_i);$$

si, de plus, $\sum_0^\infty |a_i| < \infty$, la formule est dite absolument convergente. L'auteur démontre que la classe des fonctions continues ne possède pas de formule de quadrature simple, tandis que la classe des polynomes en possède une absolument convergente pourvu que $\limsup |\lambda_i| = \infty$.

J. Favard (Paris).

Bochner, S. Closure classes originating in the theory of probability. Proc. Nat. Acad. Sci. U. S. A. **39**, 1082–1088 (1953).

A sequence of functions defined in k-dimensional Euclidean space E is said to be P-convergent, if it converges uniformly in every compact subset of E. The author investigates the P-closure of the family of functions (1) $\{\int_{|x|>\theta} K(x, y)dF(y)\}$, where K is a given kernel and F ranges over all non-negative set functions in $E-(0)$ for which the integral exists. The consideration of such families of functions is suggested by the theory of infinitely divisible laws. A typical result is Theorem 1: (a) If $K(x, y) = K_0(x, y) = \sin^{2m}(\sum_{j=1}^k x_j y_j)$, then (1) is not P-closed. The P-closure of S is the set of functions of the form (2) $\{Q(x) + \int_{|x|>\theta} K_\theta(x, y)dF(y)\}$, where Q is a sum of $2m$th powers of linear forms in the x_j. (b) The set (2) where Q runs through all non-negative homogeneous polynomials of the x_j is P-closed. (c) For a given function of the set described under (b) the polynomial Q in (2) is unique. Theorem 1 is deduced from theorems concerning the P-closure of more general sets (1).

W. H. J. Fuchs.

Bagemihl, F., and Seidel, W. A general principle involving Baire category, with applications to function theory and other fields. Proc. Nat. Acad. Sci. U. S. A. **39**, 1068–1075 (1953).

The following category theorem is established: Let S be a complete non-empty metric space. Let the class $\{P_n(s)\}$ consist of a finite or an enumerable number of properties each meaningful for every element $s \in S$ and let $Q(s)$ be the negation of their disjunction. Suppose: (1) for each P_n, if X is a dense subset of a non-empty open subset $G \subset S$ and if every $x \in X$ has the property P_n, then every $g \in G$ also has property P_n; (2) for every P_n, every non-empty open subset of S has an element which does not have property P_n. Then there exists a residual set $R \subset S$ every element of which has property Q.

Numerous applications are given. We cite the following: (a) Let $\{j\}$ denote the set of integers and $\{t_m\}$ an increasing sequence of positive real numbers with $\lim t_m = \infty$. Then there exists a residual set, R, of real numbers such that for each $r \in R$, the set $\{t_m r + j\}$ is dense in the set of reals. (b) Let $\mathfrak{M}$ denote the class of functions meromorphic in $|z|<1$, $\mathfrak{M}^*$ the class of non-constant functions in $\mathfrak{M}$, $\mathfrak{a} = \{e^{i\theta} | -\pi \leqq \theta_1 \leqq \theta \leqq \theta_2 \leqq \pi, \theta_1 < \theta_2\}$, $P \subset \mathfrak{a}$ a set of positive measure on every subarc of $\mathfrak{a}$. If (i) $f \in \mathfrak{M}$ and f has no limit along every radius terminating in a point of P, or (ii) $f \in \mathfrak{M}^*$ and the cluster set of f on every radius terminating in a point of P contains a fixed constant, finite or infinite, or (iii) $f \in \mathfrak{M}$ and P is the complement of the set of Fatou points of f, then there exists a residual set $R \subset \mathfrak{a}$ such that the radial cluster set of f corresponding to every $r \in R$ is the whole complex plane. The proof of (b) uses the theorem of Plessner and an extension of the theorem of F. and M. Riesz. This theorem and Theorem 10 of the paper contain the answer to certain function-theoretic questions proposed by P. Erdös to the authors.

M. Heins.

Freud, Géza. On a Tauberian theorem. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **3**, 45–53 (1953). (Hungarian)

The author gives a more elementary proof of a special case of a previous result of his [Acta Math. Acad. Sci. Hungar. **2**, 299–308 (1951); these Rev. **14**, 361]. In the case considered in this paper α is 1. Some of the results of the author have been obtained independently by Korevaar [Duke Math. J. **18**, 723–734 (1951); these Rev. **13**, 227] and Postnikov [Doklady Akad. Nauk SSSR (N.S.) **77**, 193–196 (1951); these Rev. **12**, 820]. *P. Erdös.*

Theory of Sets, Theory of Functions of Real Variables

Have Russian

***Alekszandrov, P. Sz., és Kolmogorov, A. N. Bevezetés a halmazelméletbe és a függvénytanba. Első rész. [Introduction to the theory of sets and the theory of functions. Part one.] = Alekszandrov, P. Sz. Bevezetés a halmazok és függvények általános elméletébe. [Introduction to the general theory of sets and functions.]** Akadémiai Kiadó, Budapest, 1952. 276 pp. 45 Ft.

Translation by Gy. Bizám of Aleksandrov's Vvedenie v obščuyu teoriyu množestv i funkcil [Gostehizdat, Moscow-Leningrad, 1948; these Rev. **12**, 682].

***Saarnio, Uuno. Die Wohlordnung einer nichtabzählbaren Menge und die Lösung des Kontinuumproblems.** Abhandlungen der Gesellschaft für Logik und ihre Anwendungen, No. 1. Helsinki, 1953. 59 pp.

A one-to-one correspondence is established between a certain subclass D of the class of all infinite subsets of the set of natural numbers and the class of all nonzero ordinal numbers less than a certain ordinal denoted in this paper by $P^{\omega}(\omega, \omega)$. The author claims that $|P^{\omega}(\omega, \omega)| = 2^{\aleph_0}$, and that, therefore, since there are at most enumerably many ordinals in the initial segment determined by any ordinal less than $P^{\omega}(\omega, \omega)$, the set D has power $2^{\aleph_0} = \aleph_1$, and has been well-ordered. The cardinal number of $P^{\omega}(\omega, \omega)$, however, is $\aleph_0$, because $P^{\omega}(\omega, \omega) = \lim_{n<\omega} P^{n}(\omega, \omega)$, where every $P^{n}(\omega, \omega)$ $(n<\omega)$ is a certain ordinal less than ω_1, and so the author has not solved the problems mentioned in the title.

F. Bagemihl (Princeton, N. J.).

***Fraïssé, Roland. Sur quelques classifications des systèmes de relations.** Thèse, Université de Paris, 1953. iii+viii+152 pp.

Le mémoire étudie des n-relations (n entier >0) dans un ensemble E et des multirelations, c.à.d. des suites vides ou finies de relations pareilles; une n-relation de base E est une application quelconque de E^n en un ensemble composé de 2 points dénoté $\{+, -\}$. Des multirelations sont un cas particulier des polyrelations [C. R. Acad. Sci. Paris **230**, 1022–1024 (1950); ces Rev. **11**, 585]. Les résultats principaux ont été annoncés [ces Rev. **10**, 517; **11**, 17, 585; **12**, 14; **13**, 99, 447, 924]; l'exposition actuelle suit un ordre presque inverse. Le mémoire peut être partagé en 2 parties: la première partie (chap. 1–6) étudie des types de relations en généralisant ce qui se passe dans le cas des relations d'ordre; la seconde partie (ch. 7–9) étudie des liens entre des polyrelations et quelques théories de logique, en particulier la méthode sémantique de Tarski [Proc. Internat. Congress Math., Cambridge, Mass., 1950, v. 1, Amer. Math. Soc., Providence, R. I., 1952, pp. 705–720; ces Rev. **13**, 521]. L'un des soins principaux consiste en classifications des relations; l'auteur introduit plusieurs espèces d'équivalences, d'ordinations, etc; on accentue, en particulier, la notion de parenté entre des multirelations et polyrelations. Soit σ une suite vide ou finie d'ordinaux $<\omega$ (signature de la multirelation). La restriction, le prolongement d'une relation, multirelation, polyrelation s'introduisent naturellement; il en est de même d'isomorphie et de type (par convention, la relation de signature vide et de base E se confond avec E, et son type est le cardinal de E). M, M' étant deux multirelations de signatures σ, σ', alors MM' désigne la multirelation obtenue par juxtaposition de M et M'; M, M' étant de même signature, p entier, l'auteur dit que M' est p-ainée de M, $M \underset{p}{>} M'$, si pour chaque $q \leqq p$ la M' restreinte à q points est isomorphe à une restriction de M. Si $M \underset{p}{>} M'$ et $M' \underset{p}{>} M$, on écrit $M \overset{1}{\underset{p}{\sim}} M'$ et l'on dit que M, M' sont $\left[\begin{smallmatrix}1\\p\end{smallmatrix}\right]$-parentes. Si $M \overset{1}{\underset{p}{\sim}} M'$ pour chaque entier p, alors M, M' sont dites 1-parentes ou $M \overset{1}{\sim} M'$. L'auteur démontre plusiers propositions en se servant des notions précédentes. On définit la χ-parenté pour $1 \leqq \chi < \omega$ et même pour $\chi = \left[\begin{smallmatrix}n\\\omega\end{smallmatrix}\right]$, p, σ étant des ordinaux $<\omega$ et σ étant une signature quelconque. Par exemple, M de base E est n-parente de M' de base E', $M \overset{n}{\approx} M'$, si pour chaque multirelation N de base E il y a une N' de base E' vérifiant $MN \overset{n-1}{\approx} M'N'$, et si cela subsiste en y permutant M, M'. Deux ensembles transfinis sont 2-parents sans être nécessairement 3-parents. L'auteur y fait introduire des filtres. On pose $\left[\begin{smallmatrix}1\\p\end{smallmatrix}\right] = \left[\begin{smallmatrix}1\\p\omega\end{smallmatrix}\right]$; pour $n>1$, on dit que M, M' sont $\left[\begin{smallmatrix}n\\p\omega\end{smallmatrix}\right]$-parentes, $M \overset{n}{\underset{p\omega}{\sim}} M'$ si, pour chaque multirelation N de base E et de signature ρ restriction de σ, il existe une N' telle que $MN \overset{n-1}{\underset{p\omega-\rho}{\sim}} M'N'$, et si l'on y peut permuter M, M'. Si $M \overset{n}{\underset{p\omega}{\sim}} M'$ quels que soient p, σ, M et M' sont n-parentes bornées, $M \overset{n}{\sim} M'$; $M \overset{\omega}{\sim} M'$ veut dire $M \overset{n}{\sim} M'$ pour chaque entier >0. La parenté est une équivalence; les classes de multirelations de signature donnée et $\left[\begin{smallmatrix}n\\p\omega\end{smallmatrix}\right]$-parentes sont en nombre fini. M est $\left[\begin{smallmatrix}n\\p\omega\end{smallmatrix}\right]$-determinable si chaque multirelation de même base et $\left[\begin{smallmatrix}n\\p\omega\end{smallmatrix}\right]$-parente de M est isomorphe de M. L étant une multirelation de base E, la $\chi - L$-classe est constituée des relations M, M' telles que LM, LM' soient χ-parentes. L'auteur énonce l'hypothèse A que voici: chaque $\left[\begin{smallmatrix}n\\p\omega\end{smallmatrix}\right] - L$-classe contient une $\left[\begin{smallmatrix}m\\q\omega\end{smallmatrix}\right] - L$-classe formée de multirelations isomorphes. Si la base E est finie, A est prouvable. L'hypothese A, pour E dénombrable, est en rapport avec quelques problèmes classiques (p. ex. l'existence d'un ensemble linéaire de puissance $\aleph_2$). L'étude des polyrelations $P = (M, x)$ (paire ordonnée d'une multirelation M et d'une suite vide ou finie x) se ramène à celle des multirelations par un certain passage moyennant lequel les notions principales concernant les M s'appliquent à des P. Les classes de polyrelations se prêtent à des considérations par lesquelles l'auteur établit un parallélisme avec ce qui se passe dans la logique mathématique. Le mémoire contenant un bon nombre de notions, théorèmes, problèmes et d'hypothèses, l'index alphabétique par lequel le travail se termine facilite considérablement la lecture de la Thèse. *G. Kurepa.*

Iseki, Kiyosi. On a proposition which is equivalent to the axiom of choice. J. Osaka Inst. Sci. Tech. Part I. **2**, 109 (1950).

Si tratta della proposizione: nessun numero cardinale infinito può esprimersi come somma di due numeri cardinali

più piccoli di lui. Dobbiamo avvertire che nel libro di W. Sierpiński, Leçons sur les nombres transfinis [Gauthier-Villars, Paris, 1928, pp. 99–100, 233], si trova lo stesso teorema dell'A. (sostanzialmente con la stessa dimostrazione), attribuito a S. Lesniewski. *T. Viola* (Roma).

Schmidt, Jürgen. Beiträge zur Filtertheorie. II. Math. Nachr. **10**, 197–232 (1953).

[Pour parti I voir Math. Nachr. **7**, 359–378 (1952); ces Rev. **14**, 255.] Considérations générales sur les relations binaires, les correspondances de Galois, le passage aux complémentaires, la relation $A \cap B \neq \emptyset$, et la notion de familles compatibles d'ensembles (deux familles de parties d'un ensemble E sont dites compatibles si tout ensemble de l'une rencontre tout ensemble de l'autre). Cette dernière notion est utilisée pour étudier les rapports des notions de filtre et de grille [Choquet, C. R. Acad. Sci. Paris **224**, 171–173 (1947); ces Rev. **8**, 458]. *P. Samuel.*

Kober, H. On quasi-decimals and on arithmetical properties of certain perfect sets and monotone functions. J. London Math. Soc. **28**, 47–58 (1953).

The author considers the unique quasi-decimal expansion $u=\sum_{n=-\infty}^{\infty} a_n \alpha^{-n}$, where $u>0$, $\alpha>1$, and the a_n are non-negative integers, and various properties of sets and functions associated with this expansion. The results are of quite complicated nature. The most important applications of sets of this type, which generalize the Cantor ternary set, are in the field of trigonometric series, in connection with sets of uniqueness, [cf. Salem, Trans. Amer. Math. Soc. **56**, 32–49 (1944); these Rev. **6**, 47] and in the theory of measure of fractional dimension. *R. Bellman.*

Arin'š, E. G. On a generalization of a theorem of Baire. Uspehi Matem. Nauk (N.S.) **8**, no. 3(55), 105–108 (1953). (Russian)

Let E be a metric space which is an absolute G_δ and has a countable open basis. The following theorems are proved. If a function f on E is the limit of an increasing sequence of upper semi-continuous functions, then the function $f(\ |P)$ is lower semi-continuous on a set of 2nd category for every perfect subset P of E. If a function f on E is Borel measurable and if $f(\ |P)$ is lower semi-continuous on a set of 2nd category for every perfect subset P of E, then f is the limit of an increasing sequence of upper semi-continuous functions on E. *E. Hewitt* (Seattle, Wash.).

Mamuzić, Zlatko. La mesure (*B*) d'un groupe d'ensembles parfaits de points du segment [0, 1]. Bull. Soc. Math. Phys. Serbie **5**, no. 1–2, 31–45 (1953). (Serbo-Croatian. French summary)

The set $P^{(s)}$ of all $x \in [0, 1]$ whose decimal expansions employ only the integers $0, 1, 2, \cdots, s (1 \leqq s \leqq 8)$ is a perfect set of measure 0. [See for example Hardy and Wright, Theory of numbers, Oxford, 1938, pp. 120–123, for the measure; the fact that $P^{(s)}$ is perfect is elementary.] Furthermore, $P^{(s)} - P^{(s-1)}$ is open in $P^{(s)}, P^{(s+1)}, \cdots, P^{(8)}$. *E. Hewitt* (Seattle, Wash.).

Sobolev, V. I. On a partially ordered measure of sets, measurable functions and certain abstract integrals. Doklady Akad. Nauk SSSR (N.S.) **91**, 23–26 (1953). (Russian)

Let X be a partially ordered ring as defined by the author in an earlier communication [same Doklady (N.S.) **56**, 237–239 (1947); these Rev. **9**, 290], and let $\{e_\lambda\}_{-\infty<\lambda<\infty}$ be a resolution of the identity in X. The usual theory of spectral measures can be extended to this context, and a countably additive function e on Borel sets of the line with values in X is defined such that $e(]\lambda, \mu[) = e_\mu - e_{\lambda+0}$ and $e(\{\lambda\}) = e_{\lambda+0} - e_\lambda$ for $-\infty < \lambda < \mu < \infty$. Furthermore, one has $e(M \cap N) = e(M)e(N)$, just as in the usual theory. Integrals $\int_{-\infty}^{\infty} f(\lambda) de_\lambda$ are defined by Lebesgue sums, etc., in the usual fashion. For $a \in X$, let $\{e_\lambda{}^a\}$ be the resolution of the identity in X such that $a = \int_{-\infty}^{\infty} \lambda de_\lambda{}^a$. Then $b \in X$ has the form $\int_{-\infty}^{\infty} f(\lambda) de_\lambda{}^a$ if and only if for every interval Δ, there exists an a-measurable set M_Δ such that $e^b(\Delta) = e^a(M_\Delta)$. *E. Hewitt* (Seattle, Wash.).

Cafiero, Federico. Sul passaggio al limite sotto il segno d'integrale per successioni d'integrali di Stieltjes-Lebesgue negli spazi astratti, con masse variabili con gli integrandi. Rend. Sem. Mat. Univ. Padova **22**, 223–245 (1953).

See the review of the long abstract of this paper printed in Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fiz. Mat. Nat. (8) **14**, 488–494 (1953); these Rev. **15**, 110. *M. M. Day.*

Phakadze, Š. S. On iterated integrals. Soobščeniya Akad. Nauk Gruzin. SSR **14**, 3–10 (1953). (Russian)

Let f be a bounded function on the unit square I^2. Define $\bar{\int}_I f(x, y)dx$ to be the upper Lebesgue integral of f with respect to x; then again use the upper L integral to define $\bar{\int}_I \bar{\int}_I f(x, y)dxdy$; similarly, define lower x, y and upper and lower y, x integrals. The strong x, y integral $\int_I{}^* \int_I{}^* f(x, y)dxdy$ is $\lim_{n\to\infty} \sum_{j \leqq 2^n} \bar{\int}_I \bar{\int}_{I_{jn}} f(x, y)dxdy$, where the I_{jn} are 2^n non-overlapping intervals whose union is I. If φ_E is the characteristic function of a subset E of I^2, then the set functions $\Phi^{*xy}(E) = \int_I{}^* \int_I{}^* \varphi_E dxdy$ and the dual $\Phi_{*xy}(E)$ are complementary outer and inner Carathéodory measures in I^2. Then f is called ordinary (strong) iterated integrable if all four ordinary (strong) iterated integrals are equal. The corollaries of Theorems 2 and 3 assert that ordinary (strong) iterated integrability of f is equivalent to existence and equality of the two iterated integrals in the usual sense of Lebesgue (over every subrectangle of I^2). Some convergence theorems are stated for these integrals, and a kind of iterated measurability is given as a sufficient condition for strong iterated integrability of f. *M. M. Day* (Urbana, Ill.).

Tung, Huai-Yuen. On Stieltjes integral of order 2. Acad. Sinica Science Record **5**, 29–43 (1952). (Chinese summary)

The Stieltjes integral of order 2, $\int_a^b f d^2\alpha$, is defined as

$$\lim_{|\Delta|\to 0} \sum_{i=1}^{n-1} f(x_i)\left(\frac{\alpha(x_{i+1})-\alpha(x_i)}{x_{i+1}-x_i} - \frac{\alpha(x_i)-\alpha(x_{i-1})}{x_i - x_{i-1}}\right),$$

where Δ is the subdivision of (a, b) by

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b,$$

and $|\Delta| = \max(x_i - x_{i-1})$. The main theorem of the paper states that this integral exists if both f and α have bounded variation of order 2, i.e. if

$$\sum_{i=1}^{n-1} \left| \frac{\alpha(x_{i+1})-\alpha(x_i)}{x_{i+1}-x_i} - \frac{\alpha(x_i)-\alpha(x_{i-1})}{x_i - x_{i-1}} \right|$$

is bounded in Δ. In accordance with a theorem of Riesz [Ann. Sci. Ecole Norm. Sup. (3) **28**, 33–62 (1911), p. 42] such a function can be written $\alpha(x) = \int_a^x \beta(t)dt + \alpha(a)$ where $\beta(t)$ is of bounded variation on (a, b). An additive property and an integration by parts theorem are derived. Further,

if $f(x)$ is continuous and $\alpha(x)$ has a derivative $\alpha'(x)$ of bounded variation, then the integral $\int f d^2\alpha$ exists as a Stieltjes integral, with respect to α' with additional terms involving the endpoints a and b. It might be remarked that the sum which is the basis of definition for $\int f d^2\alpha$, when rearranged, becomes

$$f(a)\frac{\alpha(x_1)-\alpha(a)}{x_1-a}-f(b)\frac{\alpha(b)-\alpha(x_{n-1})}{b-x_{n-1}}-\sum_{i=1}^{n}\frac{[f(x_i)-f(x_{i-1})][\alpha(x_i)-\alpha(x_{i-1})]}{x_i-x_{i-1}}$$

from which it follows that the integral exists if $\alpha'(a)$ and $\alpha'(b)$ exist and the Hellinger integral $\int_a^b d f d\alpha/dx$ exists, giving a ready method of deriving additional existence theorems, including one of which the theorem of the note is a special case. *T. H. Hildebrandt* (Ann Arbor, Mich.).

Calderon, A. P., and Mann, H. B. On the moments of stochastic integrals. Sankhyā 12, 347–350 (1953).

Let $y_n=(x_n^{(1)},\cdots,x_n^{(k)})$ be a sequence of random vectors, converging in measure to a random vector $y=(x^{(1)},\cdots,x^{(k)})$. In this paper we shall find necessary and sufficient conditions that the joint moments of $x_n^{(1)},\cdots,x_n^{(k)}$ converge to the corresponding moments of $x^{(1)},\cdots,x^{(k)}$. Our results will be applied to compute the joint moments of stochastic integrals when the joint moments of the integrands are given. [Authors' summary.] *L. J. Savage.*

McCarthy, John. An everywhere continuous nowhere differentiable function. Amer. Math. Monthly 60, 709 (1953).

Haslam-Jones, U. S. On a generalized derivative. Quart. J. Math., Oxford Ser. (2) 4, 190–197 (1953).

f: continuous function defined on the interval

$$I=[-1,+1].$$

$\mathfrak{B}_n(n\geqq 1)$: class of the functions α of bounded variation on I such that

$$\int_{-1}^{+1} t^r d\alpha(t)=0,\quad r=0,1,\cdots,n-1,\quad \int_{-1}^{+1} t^n d\alpha(t)=n!.$$

c: any number between -1 and $+1$.

$$\Delta_\alpha f(c,h)=\int_{-1}^{+1} f(c+ht)d\alpha(t)$$

for $-1\leqq c-h<c+h\leqq+1$. The "generalized derivative $D_\alpha f(c)$" is defined as $\lim_{h\to 0} h^{-n}\Delta_\alpha f(c,h)$ if this limit exists. $a_n=T_n f(c)$ is called the "nth Peano derivative" of $f(x)$ at $x=c$ if there exist constants $a_0,a_1,\cdots,a_{n-1}$ such that $f(c+h)=a_0+a_1h+a_2h^2+\cdots+a_nh^n/n!+\epsilon(h)h^n$, where $\epsilon(h)\to 0$ as $h\to 0$. It is easily shown that the existence of $T_n f(c)$ implies the existence of $D_\alpha f(c)=T_n f(c)$ for every $\alpha\in\mathfrak{B}_n$. Two conditions for α are given in order that the operator D_α be equivalent to T_n. The main purpose of the note is to show that for a suitably chosen $\alpha=\omega\in\mathfrak{B}$ the operators D_ω and T_n are equivalent. Here is the definition of ω: $p(t)=-\frac{1}{2}\{P_n(t)+P_{n-1}(t)\}$ where $P_n(t)$ is the nth Legendre polynomial, $\omega(t)=p(t)$ when $-1\leqq t<1$, $\omega(1)=1+p(1)$.

Chr. Pauc (Nantes).

Ravetz, J. The Denjoy theorem and sets of fractional dimension. J. London Math. Soc. 29, 88–96 (1954).

Construction of a continuous real function $f(x)$ $(0\leqq x\leqq 1)$ such that, in a set of x of Hausdorff dimension 1, the quantities D_+f, D_-f, D^-f vanish, whilst D^+f is positively infinite. *L. C. Young* (Madison, Wis.).

Kuttner, B. Some theorems on fractional derivatives. Proc. London Math. Soc. (3) 3, 480–497 (1953).

Let $I_+^\kappa f$, $I_-^\kappa f$, $D_+^\kappa f$, $D_-^\kappa f$ be the forward and backward fractional integrals and derivatives of an integrable $f(x)$. The author is interested in connections between properties of a forward and a backward integral or derivative. Most of the results refer to the case $0<\kappa<1$, there being minor modifications in the general case. If $0<\kappa<1$, and $f(x)=I_+^\kappa\phi(x)$, then $D_+^\kappa f(x)$ exists almost everywhere; but at almost all points, $D_-^\kappa f(x)$ exists if and only if $(d/dx)\int\phi(u)\log|x-u|du$ exists (and hence may fail to exist anywhere). However, if $\phi(x)$ satisfies a Lipschitz condition or integrated Lipschitz condition, $D_-^\kappa f(x)$ generally exists and satisfies a similar condition; the precise statements are rather delicate and vary according to whether the whole of the basic interval or only an interior subinterval is considered. *R. P. Boas, Jr.*

Giuliano, Landolino. Sopra un'estensione del concetto di funzione generalmente a variazione limitata. Ann. Scuola Norm. Super. Pisa (3) 7, 65–78 (1953).

Let $\phi(u)$ increase continuously with u from 0 to ∞ and let $L(\phi)$ denote the class of real functions $g(t)$, measurable and almost everywhere finite in $0\leqq t\leqq 1$, for which $\phi(|g|)$ is integrable. A real function $f(x,y)$, measurable in the unit square Q of the (x,y)-plane, is termed by the author ϕ-generally of bounded variation, if for some subset E of Q of plane measure zero, the total variations, outside E, of $f(x,y)$ on the lines $x=t$ and $y=t$, belong to $L(\phi)$. The author observes that several theorems known in the case $\phi(u)=u$ remain valid. For instance, if $f(x,y)$ is ϕ-generally of bounded variation, then Q is the sum of a set of plane measure zero and a countable number of sets in each of which $f(x,y)$ is Lipschitzian. The details are given mainly in the case in which $\phi(u)$ is a fractional power of U, and consist of appropriate modifications of known arguments.

L. C. Young (Madison, Wis.).

Darbo, Gabriele. La nozione di variazione limitata e di assoluta continuità super-uniforme. Rend. Sem. Mat. Univ. Padova 22, 246–250 (1953).

A set of functions $[f(x)]$ is said to be super uniformly of bounded variation (B.V.s.u.) on (a,b) if there exists an M such that for every finite set of disjoint intervals (ξ_i',ξ_i'') of (a,b) and every finite subset of functions $f_i(x)$, we have $\sum_i|f_i(\xi_i')-f_i(\xi_i'')|\leqq M$. The super uniform absolute continuity (A.C.s.u.) is similarly defined. A necessary and sufficient condition for B.V.s.u. (A.C.s.u.) is that there exists a non-decreasing (A.C.) function $F(x)$ such that the variation of f on (x_1,x_2) does not exceed $F(x_2)-F(x_1)$, for every f and every $a\leqq x_1<x_2\leqq b$. Obviously the convergence theorem on a sequence of Lebesgue integrals $\int f_n(x)dx$, with $f_n(x)$ convergent a.e. holds if $\int_a^x f_n(x)dx$ are A.C.s.u.

T. H. Hildebrandt (Ann Arbor, Mich.).

Marx, Imanuel, and Piranian, George. Lipschitz functions of continuous functions. Pacific J. Math. 3, 447–459 (1953).

For any value of θ between 0 and 1, Loud [Proc. Amer. Math. Soc. 2, 358–360 (1951); these Rev. 13, 218] constructed a real continuous non-differentiable function whose behavior at every point is similar to the behavior of the function x^θ at the origin. Specifically, he proved the following theorem. Theorem 1. If θ is a constant $(0<\theta<1)$, there exist a continuous function $f(x)$ and a pair of positive constants K_1 and K_2 such that $|f(x+h)-f(x)|<K_1|h|^\theta$ for all x and

all h, and such that $\limsup_{h\to 0}[|f(x+h)-f(x)|]/|h|^{\theta}>K_x$ for all x.

The authors generalize this result by considering the "point Lipschitz function" of a given continuous function. Let $f(x)$ be a continuous real-valued function of x, and consider the quantity $Q(\theta, x; f)=\limsup_{h\to 0}[|f(x+h)-f(x)|]/|h|^{\theta}$. If Q is finite for some value of θ, it is zero for all smaller values of θ; if Q is greater than zero for some value of θ, it has the value $+\infty$ for all larger values of θ. Let $\theta(x; f)$ denote the least upper bound of all θ for which Q is finite. Then $\theta(x; f)$ is called the point Lipschitz function of $f(x)$. It can easily be seen that $\theta(x; f)=\lim\inf_{h\to 0}[\log|f(x+h)-f(x)|]/\log|h|$. It follows almost immediately from this that $\theta(x; f)$ has the following properties:

Theorem 2. If $f(x)$ is continuous $(-\infty<x<+\infty)$, then $\theta(x; f)$ is the inferior limit of a sequence of continuous functions of x; if $\theta(x; f)>1$ throughout some open interval of the x-axis, then $\theta(x; f)=+\infty$ throughout the interval.

The result of Loud is included in the following theorem which, together with Theorem 2, provides a characterization of those point Lipschitz functions of continuous functions which lie everywhere between 0 and 1 inclusive. Theorem 3. Let $\{\theta_n(x)\}$ be a sequence of continuous functions $(0\leqq\theta_n(x)\leqq 1)$, and let $\theta(x)=\lim\inf_{n\to\infty}\theta_n(x)$. Then there exists a continuous function $f(x)$ such that $\theta(x; f)=\theta(x)$.

The authors also consider the local Lipschitz function of a continuous function, defined in a similar manner in terms of the difference quotient $[|f(x_2)-f(x_1)|]/(x_2-x_1)^{\theta}$, where, for fixed x and h, $x-h\leqq x_1<x_2\leqq x+h$. It is lower semi-continuous, and any lower semi-continuous function between 0 and 1 inclusive is the local Lipschitz function of a strictly increasing (hence almost everywhere differentiable) function. *B. Lepson* (Washington, D. C.).

Nagumo, Mitio. On the independence of continuous functions. J. Osaka Inst. Sci. Tech. Part I. **2**, 89–90 (1950). (Esperanto)

If D and E are topological spaces and f is a continuous map of D into E, then f will be called independent on D (relative to E) provided fD is of second category in E, and independent at the point $p\in D$ provided $f|U$ is independent on U for every open $U\ni p$. The author's principle results are the following: (1) Suppose D is separable metric, f is independent on D, and D_0 is the set of all points of D at which f is independent. Then D_0 is closed and non-empty, and $f|D_0$ is independent at each point of D_0. (2) Suppose D is metric, f is independent at each point of D, and H is the set of all points $p\in D$ such that fp is interior to the closure of fU for every open $U\ni p$. Then H is a residual G_δ set in D. (The theorems are stated less generally by the author, for he is concerned primarily with generalizing the notion of independence due to Knopp and Schmidt [Math. Z. **25**, 373–381 (1926)]. He works with m real-valued continuous functions $f_1, \cdots, f_m$ on D, and this family is called independent provided the associated map f of D into $E^m(fx=(f_1x, \cdots, f_mx)\in E^m)$ is independent in the above sense.) *V. L. Klee* (Seattle, Wash.).

Ungar, Peter. Freak theorem about functions on a sphere. J. London Math. Soc. **29**, 100–103 (1954).

The author studies the following problem. Let f be a continuous function on the surface of a sphere such that $\int_\Sigma f d\sigma=0$ for all spherical caps Σ of fixed radius α. Does it follow that $f\equiv 0$? The author finds that the set of α for which the answer is affirmative (negative) is dense in $(0, \pi/2)$. The problem is analogous to one proposed by Pompeiu [Bull. Sci. Math. (2) **53**, 328–332 (1929)]. *M. O. Reade* (Ann Arbor, Mich.).

Dubovickiĭ, A. Ya. On differentiable mappings of an n-dimensional cube into a k-dimensional cube. Mat. Sbornik N.S. **32(74)**, 443–464 (1953). (Russian)

Let $u: u_i=u_i(x_1, \cdots, x_n)$ $(i=1, \cdots, k)$ be a mapping of an n-dimensional cube C_n into a k-dimensional cube C_k. The mapping is termed m-fold differentiable if at every point of C_n the functions u_i have differentials of all orders through the mth. A point at which the Jacobian matrix of the mapping degenerates is termed a singular point; the set of all singular points is denoted by Ω. Kronrod and Landis [Doklady Akad. Nauk SSSR (N.S.) **58**, 1269–1272 (1947); these Rev. **9**, 339] proved that, if $k=1$, $u(\Omega)$ has measure zero if the mapping is n-fold differentiable, but $u(\Omega)$ can be an interval if the mapping is only $(n-1)$-fold differentiable. This is generalized as follows: If the mapping u is $(n-k+1)$-fold differentiable, then $u(\Omega)$ has k-dimensional measure zero; there exists an $(n-k)$-fold differentiable mapping u for which $u(\Omega)=C_k$. It is also shown that, for almost all y in C_k, the set $u^{-1}(y)$ has a finite number of components which are $(n-k+1)$-fold differentiable manifolds. Further, if u is $(n-k+1-p)$-fold differentiable, then for almost all y in C_k, $\Omega\cap u^{-1}(y)$ has dimension at most $p-1$ and, in the conclusion, $p-1$ is best possible. The proofs, which are relatively complicated, use results of the paper of Kronrod and Landis referred to, as well as results of Whitney [Trans. Amer. Math. Soc. **36**, 63–89 (1934)] and Lyapunov [Doklady Akad. Nauk SSSR (N.S.) **65**, 609–612 (1949); these Rev. **10**, 518]. *W. Kaplan* (Ann Arbor, Mich.).

Kudryavcev, L. D. On summability of Jacobians. Mat. Sbornik N.S. **33(75)**, 389–398 (1953). (Russian)

Let G be a domain of Euclidean n-space R and let f be a differentiable mapping of G into R. Further let J be the Jacobian of f and let G_0 be the set of the points of G at which $J(x)=0$. The mapping is (weakly) monotone if the set $f^{-1}(y)$ is connected for each y in $f(G)$, compact if this set is compact. The author terms pseudo-multiplicity of a subset E of G, the smallest cardinal not exceeded by that of the intersection of E with $f^{-1}(y)$ for any y in $f(E)-f(G_0)$. Two theorems are proved: (I) if f is monotone and E is a measurable subset of G, then the measure of $f(E)$ is the integral over E of $|J|$ and (II) if f is compact and E is a compact subset of G, then the pseudo-multiplicity of E is finite and J is summable over E. The author refers in one place to Radó and Reichelderfer [Proc. Nat. Acad. Sci. U. S. A. **35**, 678–681 (1949); these Rev. **11**, 588] but makes no attempt to attain a comparable degree of generality. *L. C. Young* (Madison, Wis.).

Lodigiani, Bruna. Sulla differenziabilità asintotica regolare delle funzioni di più variabili. Rend. Sem. Mat. Univ. Padova **22**, 251–257 (1953).

Let $f(x_1, \cdots, x_n)$ be a real-valued, continuous function in the unit cube Q^n: $0\leqq x_j\leqq 1$, $j=1, \cdots, n$, in Euclidean n-space. Certain basic theorems concerning the existence of a total differential of f (in the sense of Stolz or in the "approximate" sense of Stepanoff) are known to hold for all values of n. For $n=1$, such theorems are trivial consequences of the definition of the derivative of a function of a single real variable. For $n>1$, the proofs are elaborate, and have been presented in the literature explicitly only for the case $n=2$, since it is obvious that the general case can be treated

by a cumbersome but straightforward method of induction with respect to n. However, there is a theorem concerning total differentials which is true for $n=2$ and is generally false for $n>2$, and the present note is concerned with this particular theorem. Given f as above, let $P_0=(x_1^0, \cdots, x_n^0)$ be an interior point of Q^n. Then f is said to possess a regular approximate differential ("differenziale asintotico regolare") at P_0 if there exists a measurable set E and a system of constants $a_1, \cdots, a_n$ such that the following holds. (i) P_0 is a point of density of E. (ii) E is the union of the frontiers of a family of n-cubes with center at P_0. (iii) If $P=(x_1, \cdots, x_n)$ denotes a generic point of Q^n and r denotes the distance of PP_0, then

$$\left[f(P)-f(P_0)-\sum_{j=1}^{n} a_j(x_j-x_j^0)\right]\Big/ r \to 0$$

if $P \in E \cap Q^n$ and $P \to P_0$. For $n=2$, it is true that if the first partial derivatives of f exist almost everywhere in Q^n, then f possesses a regular approximate differential almost everywhere in Q^n. This theorem, which was discovered independently by R. Caccioppoli, G. Scorza Dragoni and the reviewer about 1938, plays an important role in surface area theory and its applications, and therefore it is a regrettable fact that this theorem is generally false for $n>2$. The present paper is a contribution to the problem of determining the precise conditions for the existence of a regular approximate differential in the case $n>2$. The author considers explicitly the case $n=3$. His principal result is as follows: If $f(x, y, z)$ has first partial derivatives in the unit cube Q^3 in 3-space which are continuous with respect to each pair of the variables x, y, z, then f possesses a regular approximate differential almost everywhere in Q^3.

T. Radó (Columbus, Ohio).

Theory of Functions of Complex Variables

***Milloux, Henri. Principes, méthodes générales. Fascicule I.** Avec la collaboration de Charles Pisot. Traité de théorie des fonctions. Tome I. Gauthier-Villars, Paris, 1953. viii+300 pp. 4500 francs.

This is the first part of the first volume of a Treatise on the theory of functions of a complex variable. The title is too modest; the book goes at least as far as most textbooks of the Theory of functions. Chapter I. A summary of topological and real-variable prerequisites. These consist of the simplest notions of point-set theory and a knowledge of advanced calculus. The Stieltjes integral and the Lebesgue integral are included, but they are not really used in the present volume. Chapter II. The notion of analyticity. Harmonic functions. Conformal representation, discussion of the conformal mapping defined by some elementary functions; the hyperbolic geometry connected with the linear transformations of the unit circle into itself. Simple examples of Riemann surfaces. Chapter III. Cauchy's theorem. (For the boundary of a domain D, $f(z)$ regular in D, continuous in $\bar{D}$.) Cauchy's integral. Schwarz's lemma. Liouville's theorem. Applications of Schwarz's lemma. (Julia-Carathéodory; functions bounded by m on part of the boundary $M(>m)$ on the remainder.) Chapter IV. Isolated singularities. Weierstrass theorem. Zeros of regular functions. Rouché's theorem. Behavior at ∞. Riemann sphere. Poisson-Jensen formula. Blaschke's theorem. Proof of the convergence of the Fourier series of a function $\varphi(t)$ of bounded variation by integration of

$$F(z)=\int_t^{2\pi} e^{-z(u-t)}\varphi(u)du/(1-e^{-2\pi z})$$

round a large circle. Chapter V. Vitali's selection principle and Riemann's mapping theorem. Chapter VI. Taylor series, Laurent series, analytic continuation, Bürman-Lagrange theorem, over-convergence. Laplace transform. Indicator diagram, Borel summability. Dirichlet series. Infinite products and Hadamard theory of entire functions. Chapter VII. Integrals of analytic functions, especially hyperelliptic and elliptic integrals. Schwarz symmetry principle. Mapping of polygons. The modular function. Picard, Schottky, Landau theorems. Julia directions. Bloch's theorem.

The presentation is clear and straightforward. No previous knowledge is assumed, but it seems likely that the lack of exercises and the rapid progress of the exposition would present great difficulties to a beginner. But at the very least this book should become an extremely useful reference for additional reading. A particularly praiseworthy feature is the nice balance between geometrical and analytical methods. The author has a distinct preference for "pure function-theoretical" methods. In some instances this leads to deviations from the usual routine of exposition; an example is the very late introduction of Taylor and Laurent series, long after the discussion of isolated singularities. This does not seem advantageous to the reviewer. The discussion of many problems is pushed much further than in most elementary texts (e.g., Schwarz symmetry principle, Picard theorem, functions vanishing on an arc of the boundary). In some cases it might have economized space and effort for the treatise as a whole, if the discussion had been extended still further, e.g., by including Lindelöf's principle of sub-ordination which is used in several special cases or by giving Carathéodory's results on the Schwarz symmetry principle. But on the whole the author has steered a very good course avoiding both the dangers of too much detail and of too little content. *W. H. J. Fuchs.*

Montel, Paul. Sur un critère de normalité. C. R. Acad. Sci. Paris **237**, 536–537 (1953).

It is shown that a family F of functions $f(z)$ which are meromorphic in a domain D is normal in the interior of D if and only if, for each domain D' interior to D, the roots of the equations $f(z)=a$, $f(z)=b$, $f(z)=c$, $f(z)=d$, where a, b, c, d are different numbers, are such that the distance of each root of any one of these equations from each root of any other one is greater than a fixed positive number which is independent of the member of the family. The result is not valid for less than four values a, b, c, d, but the four values might vary with the function $f(z)$ provided only that no two of them have a common limit value. Other results of a similar nature are given. *E. F. Beckenbach.*

Pastidès, Nicolas. Sur les fonctions à centre. C. R. Acad. Sci. Paris **237**, 957–959 (1953).

Let $f(z)=az+\sum_2^\infty a_n z^n$ be a regular function of z, $a=e^{2\pi i\alpha}$, α irrational. The origin is a center of $f(z)$ if there is a regular function $\varphi(z)=z+\sum_{n=2}^\infty b_n z^n$ satisfying $f(\varphi(z))=\varphi(az)$. Formal substitution yields recursion formulae (1) $(a^n-a)b_n$ = polynomial in $a_k, b_k (k<n)$. Theorem 1. If there is a $\rho=\rho(\epsilon)$ such that in (1) $|b_n|<\rho^n$ whenever $|a^n-a|<\epsilon<1$, then O is a center of $f(z)$. Let $f_1(z)=f(z)$, $f_k(z)=f(f_{k-1}(z))$. Theorem 2. If $\varphi(z)$ is the formal power-series whose coeffi-

cients are defined by (1) and if $g(z)$ is its formal inverse power series, then the coefficients of the power series

$$h_n(z)=(z+a^{-1}f_1(z)+a^{-2}f_2(z)+\cdots+a^{-n}f_n(z))/(n+1)$$

tend term-for-term to the corresponding coefficients of $g(z)$. Corollary. If $\{f_k(z)\}$ is a normal family, then the origin is a center of $f(z)$. *W. H. J. Fuchs* (Ithaca, N. Y.).

Iliev, Lyubomir. Analytically noncontinuable series of Faber polynomials. Bŭlgar. Akad. Nauk. Izvestiya Mat. Inst. **1**, 35–56 (1953). (Bulgarian)

In the completed z-plane, let G be a simply connected region containing the point $z=\infty$. Let the function $w=\Phi(z)$, with the expansion $\Phi(z)=z+\beta_0+\beta_1/z+\cdots$ in the neighborhood of ∞, map G conformally and schlicht upon a region $|w|>\rho$ (it is assumed that $\rho<1$); and let the sequence $\{\Phi_n(z)\}$ be defined by the equation

$$[\Phi(z)]^n=\Phi_n(z)+\beta_{n1}/z+\beta_{n2}/z^2+\cdots.$$

For any sequence $c=\{c_n\}$ with $\limsup |c_n|^{1/n}=1$, the function $f(z)=\sum c_n\Phi_n(z)$ is regular in the region interior to the curve C which is the image of the circle $|z|=1$ under the mapping $w=\Phi(z)$. Elsewhere [Doklady Akad. Nauk SSSR (N.S.) **90**, 499–502 (1953); these Rev. **15**, 23], the author has shown that if c belongs to the class NP of sequences whose elements take only a finite number of different values and which are not ultimately periodic, then C is a natural boundary for $f(z)$. He now extends this result in various ways. For example, C is a natural boundary under each of the following conditions: (1) $c_n=a_n+\epsilon_n$, where $a\in NP$ and $\epsilon_n\to 0$; (2) $c_n=a_nb_n$, where $a\in NP$, $b_{n+1}/b_n\to 1$, and $b_n=O(n^\alpha)$ for some arbitrary constant α; (3) c is bounded; and for some increasing sequence $\{n_k\}$, $c_{n_k}\to\alpha$ while, for each fixed positive integer r, $c_{n_k-r}\to\beta\neq\alpha$; (4) the elements of c take only the values $d_1, d_2, \cdots, d_k$, except for a subsequence $\{c_{n_m}\}$ which is bounded, is bounded away from each d_j, and for which $\limsup (n_m-n_{m-1})=\infty$. *G. Piranian.*

Polak, A. Analytic functions and open mappings. Doklady Akad. Nauk SSSR (N.S.) **89**, 617–618 (1953). (Russian)

The following theorem is proved: Let $w=f(z)$ be analytic in a domain D of the z-plane and let $f(z)=w_0$ only for $z=z_0$. Let a be a positive number such that the set F for which $|f(z)-w_0|\leq a$ is a continuum and let d be the greatest distance between points of F at which $f(z)$ takes equal values. Then for every d', $0<d'<d$, there exists a w for which $|w-w_0|\leq a$ and the maximum distance between points of F at which $f(z)=w$ equals d'. The theorem is deduced from a general theorem of the author on open mappings [C. R. (Doklady) Acad. Sci. URSS (N.S.) **1**, 155–157 (1936)]. *W. Kaplan* (Ann Arbor, Mich.).

Hersch, Joseph, et Pfluger, Albert. Principe de l'augmentation des longueurs extrémales. C. R. Acad. Sci. Paris **237**, 1205–1207 (1953).

Let $\{C_1\}, \{C_2\}, \cdots, \{C_k\}$ be families of curves in the z-plane. The authors modify slightly an earlier definition of Jenkins [Trans. Amer. Math. Soc. **67**, 327–350 (1949); these Rev. **11**, 341] and define

$$(1)\qquad M=M(a_1, a_2, \cdots a_k)=\inf_\rho \overline{\int\int} \rho^2 d\tau,$$

where ρ is a positive function defined on the curve families such that

$$(2)\qquad \int_{C_r}\rho|dz|\geq a_r>0,\quad r=1, \cdots, k,$$

where the upper and lower Darboux integrals are taken in (1) and (2) respectively. They prove the principle that if $z'=f(z)$ is analytic on the curve families and M' is defined with the same constants a_r, with respect to the image curves $\{C_r'\}$, then $M'\leq M$. In particular, the extremal length [Ahlfors and Beurling, Acta Math. **83**, 101–129 (1950), p. 114; these Rev. **12**, 171] of any given curve family is not decreased by an analytic transformation.

This result is related to, but more general and precise than, certain earlier theorems of Jenkins [Contributions to the theory of Riemann surfaces, Princeton, 1953, pp. 87–94; these Rev. **15**, 115] and the authors deduce two results obtained there from their principle. *W. K. Hayman.*

Mikusiński, J. G.-. On the Paley-Wiener theorem. Studia Math. **13**, 287–295 (1953).

The author proves that if $F(z)$ is analytic and of exponential type k in $x>0$ with boundary values $F(iy)$, and if $F(iy)\in L^p$, $1\leq p\leq 2$, then $F(z)$ is represented by the absolutely convergent integral $\int_{-k}^{\infty}e^{-zt}f(t)dt$. Here $f(t)$ is the Fourier transform of $F(iy)$ (in the appropriate sense) and vanishes almost everywhere for $t<-k$. The author's proof is of interest, but the theorem itself seems to be implicit in work of Paley and Wiener [Fourier transforms in the complex domain, Amer. Math. Soc. Colloquium Publ. vol. 19, New York, 1934, p. 8, for $p=2$] and Hille and Tamarkin [Fund. Math. **25**, 329–352 (1935), p. 345]. *R. P. Boas, Jr.* (Evanston, Ill.).

Leont'ev, A. F. On representation of entire functions by sequences of linear aggregates. Mat. Sbornik N.S. **33(75)**, 453–462 (1953). (Russian)

If $\{f_n(z)\}$ is a sequence of entire functions of finite order, the author says [following Pólya] that the sequence is of finite order if $|f_n(z)|\leq\exp\{|z|^\mu\}$ for all n and $|z|>r_0$ (independent of n); the infimum of such μ is called the order of the sequence. Let $f(z)$ be an entire function of order ρ with no $f^{(n)}(0)=0$, and let $\{\lambda_n\}$ be a sequence of distinct complex numbers of increasing absolute value with exponent of convergence $\rho_1>0$. Let $F(z)$ be an entire function of order ν. The author proves that there is a sequence of finite sums $f_n(z)=\sum a_{nj}f(\lambda_j z)$ converging everywhere to $F(z)$ and having order at most $\max\{\nu, \rho\rho_1/(\rho_1-\rho)\}$. The novelty of the result consists in the fact that a bound is obtained for the order of the approximating sums. That the bound cannot generally be reduced follows from the complementary result that if $f(z)=\sum a_n z^n$ and $\lim (n\log n)/\log|1/a_n|=\rho$, and the other conditions of the first theorem are satisfied, then every entire function $F(z)$, of order less than ρ, which is a limit of $f_n(z)$ of the indicated form, with the order of $\{f_n(z)\}$ less than $\rho\rho_1/(\rho_1-\rho)$, is identically 0.

R. P. Boas, Jr. (Evanston, Ill.).

Zmorovič, V. A. On certain variational problems of the theory of univalent functions. Ukrain. Mat. Žurnal **4**, 276–298 (1952). (Russian)

The author proves two general theorems on the extreme values for certain functionals defined for classes of analytic functions generated by Stieltjes integrals. These theorems are then applied to determine the maximum and the minimum curvature $K(r)$ of the image of $|z|=r<1$, (a) for the class of normalized univalent functions mapping $|z|<1$ onto a convex region, (b) for the subclass of (a) with s-fold symmetry, (c) for the class of normalized univalent functions mapping $|z|>1$ onto the complement of a convex region. Further, the extreme values of $\arg f'(z)$ and $|f'(z)|$

are obtained for the class of normalized univalent functions having bounded rotation. For (a) the result is

$$\frac{1-r^2}{r}\leqq K(r)\leqq\frac{1-r^2}{r}\frac{2}{t}\sinh\frac{t}{2}\exp\left(-1+\frac{t}{2}+\frac{t}{e^t-1}\right),$$

where $t=2\ln((1+r)/(1-r))$. Equality occurs on the right at $z=r$ for the function

$$f(z)=[1-((1-z)/(1+z))^{1-2\lambda}]/(2-4\lambda)$$

which maps $|z|<1$ onto a sector with vertex angle $(1-2\lambda)\pi$, where $0\leqq\lambda=t^{-1}-(e^t-1)^{-1}\leqq\frac{1}{2}$, and equality occurs on the left for the same function at $z=ir$. This result is noteworthy because for the upper bound the extremal function changes with r, while for the lower bound all values of λ, $0\leqq\lambda\leqq\frac{1}{2}$ yield functions for which equality is attained on the right, so that the extremal function is not unique apart from rotations.

The author further considers certain other classes of functions which are univalent by a theorem discovered independently by Noshiro [J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. 2, 129–155 (1934)] and Warschawski [Trans. Amer. Math. Soc. 38, 310–340 (1935)].

A. W. Goodman (Lexington, Ky.).

Keogh, F. R. Some inequalities for convex and star-shaped domains. J. London Math. Soc. 29, 121–123 (1954).

Let $\sigma(z)=z+\cdots$ be univalent in $|z|<1$, let D be the image region and let C_r of length L_r be the image curve of $|z|=r$. It is proved that if D is convex then (a) the radius of curvature of $C_r\leqq r/(1-r^2)$, and (b) $L_r\leqq 2\pi r/(1-r^2)$. If D is star-shaped with respect to the origin then

(c) $$\Re((z\sigma'(z)/\sigma(z))\geqq|\sigma(z)|(1-r^2)/r,$$

and

(d) $$\int_0^{2\pi}(re^{i\theta})d\theta\leqq 2\pi r/(1-r^2).$$

All the inequalities are sharp. The bound in (a) was obtained independently by Zmorovič [see the paper reviewed above]. The proof given by Zmorovič is much more complicated than the author's, but is more revealing.

A. W. Goodman (Lexington, Ky.).

Tammi, Olli. On the coefficients of the solutions of Löwner's differential equation. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 148, 7 pp. (1953).

Let $w=F(z,u)=z+\sum_{n=2}^{\infty}a_n(u;\theta(u))z^n$ be the solution of Löwner's differential equation

$$\frac{\partial F(z,u)}{\partial u}=2F(z,u)\frac{e^{i\theta(u)}F(z,u)}{1-ue^{i\theta(u)}F(z,u)},\quad 0<x\leqq u\leqq 1,$$

for which $F(z,1)=z$, $F(0,u)=0$, $F_z'(0,u)=1$. Then $F(z,u)$ is schlicht and bounded, $|F(z,u)|<x^{-1}$ in $|z|<1$. Let the coefficients of the inverse function $\phi(w,x)$ of $F(z,x)$ be denoted by $a_n(x;\theta(u))$. Then

$$a_n(x;\theta(u))=x^{n-1}a_n(x^{-1};\theta(xu)),$$
$$a_n(x;\theta(u))=x^{n-1}a_n(x^{-1};\theta(xu))$$

for $n=2,3,\cdots$.

M. S. Robertson.

Tammi, Olli. On the maximalization of the coefficient a_3 of bounded schlicht functions. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 149, 14 pp. (1953).

Let $F(z,x)=z+\sum_{n=2}^{\infty}a_n(x)z^n$ be the solution of the Löwner differential equation, mapping the unit circle into a circle of radius x^{-1} with a Jordan curve as slit. The third coefficient

(1) $$a_3(x)=4\left(\int_1^x e^{i\theta(u)}du\right)^2-2\int_x^1 ue^{i\theta(u)}du,$$

where $\theta(u)$ is a real continuous function appearing in the differential equation. The author replaces $\theta(u)$ by a step-function $\theta_\nu(u)$ and obtains for the corresponding third coefficient

(2) $$a_3^*(x)=\frac{(1-x)^2}{(N/2)^2}\left(\sum_{\nu=1}^{N}e^{i\theta_\nu}\right)^2-\frac{1-x}{N}\sum_{\nu=1}^{N}\left\{2x+\frac{2\nu-1}{N}(1-x)\right\}e^{i\theta_\nu}.$$

By maximizing $a_3^*(x)$ and letting $N\to\infty$ the author solves the problem of maximizing $a_3(x)$. If

$$f(z)=z+\sum_{n=2}^{\infty}a_n(x)z^n$$

maps the unit circle $|z|<1$ into a schlicht domain G, the boundary of which has a maximal distance $1/x$ from the origin, then

(3) $$|a_3(x)|\leqq 1-x^2\quad\text{for}\quad e^{-1}\leqq x\leqq 1.$$

The equality sign in (3) is valid only when $\theta(u)$, with the exception of an arbitrary additive constant, is defined by the equations

$$\cos\theta(u)=0,\quad\int_x^1\sin\theta(u)du=0,\quad x\leqq u\leqq 1,$$

(4) $$|a_3(x)|\leqq 2(\sigma-x)^2+1-x^2\leqq 3\quad\text{for}\quad 0\leqq x\leqq e^{-1},$$
$$\sigma\log\sigma+x=0,\quad e^{-1}\leqq\sigma\leqq 1.$$

The equality sign in (4) holds only when $\theta(u)$ (except for an additive constant) is defined by $\theta(u)=0$, $x\leqq u\leqq\sigma$, $\cos\theta(u)=\sigma/u$, $\sigma\leqq u\leqq 1$, and $\int_\sigma^1\sin\theta(u)du=0$. Analogous results are obtained for the maximums of the combinations $2a_3(x)-a_2(x)^2$ and $6a_3(x)-4a_2(x)^2$.

M. S. Robertson.

Umezawa, Toshio. On the coefficients of multivalent functions. J. Math. Soc. Japan 5, 137–144 (1953).

Let $w=f(z)=a_1z+a_2z^2+\cdots+a_nz^n+\cdots$ be regular for $|z|\leqq 1$ and map $|z|=1$ into a curve C with the property that there exists a straight line through the origin which cuts C in exactly $2p$ points. If the straight line is the real axis and the coefficients a_n are all real, the following inequalities [see Goodman and Robertson, Trans. Amer. Math. Soc. 70, 127–136 (1951); these Rev. 12, 691] are known to be sharp:

(1) $$|a_n|\leqq\sum_{k=1}^{p}\frac{2k(n+p)!}{(p+k)!(p-k)!(n-p-1)!(n^2-k^2)}|a_k|.$$

If the coefficients are complex, (1) must be replaced by the sharp result [see Robertson, Canadian J. Math. 4, 407–423 (1952); these Rev. 14, 460]

(2) $$|a_n|\leqq\sum_{k=1}^{p}\frac{2n(n+p)!}{(p+k)!(p-k)!(n-p-1)!(n^2-k^2)}|a_k|.$$

The author considers the case where the coefficients are complex and the given straight line coincides with the diametral line. For this special case, in attempting to show that (1) holds, he finds instead that for $n=p+1$

(3) $$|a_{p+1}|\leqq\sum_{k=1}^{p}\frac{2(2p)!(p^2+p+k^2)}{(p+k+1)!(p-k+1)!}|a_k|,$$

but does not claim that (3) is sharp. Further inequalities are obtained for the coefficients of functions $f(z)$ on which special symmetrical conditions have been imposed.

M. S. Robertson (New Brunswick, N. J.).

Sunyer Balaguer, F. Number of Borel directions and exceptional values of a meromorphic function of finite order. Mem. Real Acad. Ci. Art. Barcelona **30**, 451–459 (1952). (Spanish)

The work of Cartwright on integral functions of integer order [Proc. London Math. Soc. **33**, 209–224 (1931)] is supplemented by giving precise values to certain constants. Let $f(z)$ be a meromorphic function of order ρ and proximate order $\rho^{(r)}$. If $\limsup n(r,a)/r^{\rho^{(r)}}<B$ for two values of a where B depends only on ρ, it is shown that $f(z)$ has at least 2ρ directions of Borel of maximum type.

A. J. Macintyre (Aberdeen).

Dinghas, Alexander. Zu Nevanlinna's zweitem Fundamentalsatz in der Theorie der meromorphen Funktionen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **151**, 8 pp. (1953).

A revision of the theory leading to the inequalities of Nevanlinna and Ahlfors

$$\sum_1^q N(r,a_n)-\sum_1^q N_1(r,a_n)\geqq(q-2)T(r)-O(S_1(r))$$

$$\sum_1^q n(r,a_k)-\sum_1^q n_1(r,a_k)\geqq(q-2)S(r)+O(L(r))$$

[R. Nevanlinna, Eindeutige analytische Funktionen, Springer, Berlin, 1st ed., 1936, 2nd ed., 1953, pp. 237–286; these Rev. **15**, 208]. The account is simpler than usual, avoids the theory of covering surfaces but gives a modified remainder term in the second formula. *A. J. Macintyre.*

Sz.-Nagy, Gyula. Über die nichtreellen Werte einer totalreellen rationalen Funktion. Acta Math. Acad. Sci. Hungar. **4**, 89–94 (1953). (Russian summary)

Let $a_1<b_1<a_2<b_2<\cdots<a_n<b_n$, and let K_0 be the circle with diameter (a_1, b_n), and K_k be the circle with diameter (a_k, b_n), $k=1, 2, \cdots, n$. Then for the function

$$F(z)=\prod_{k=1}^{n}(z-b_k)/(z-a_k),$$

$\Re F>0$ for z outside K_0, and $\Re F<0$ for z inside K_k, $k=1, 2, \cdots, n$. Various extensions of this theorem are indicated: thus, if $\sum_{k=1}^{\infty}a_k^{-2}<\infty$ and $0<a_k<b_k<a_{k+1}$, $k=1, 2, \cdots$, then for the function

$$G(z)=\prod_{k=1}^{\infty}\left(\frac{a_k}{b_k}\right)^2\frac{z^2-b_k^2}{z^2-a_k^2}$$

$\Im G(z)>0$ for z in quadrants I and III, and $\Im G(z)<0$ for z in quadrants II and IV. *A. W. Goodman.*

Flett, T. M. Note on a function-theoretic identity. J. London Math. Soc. **29**, 115–118 (1954).

Let $g(R)$ be absolutely continuous for $R\geqq0$ and let $G(R)=\int g(R)d(\log R)$. Let $f(z)$ be regular in $|z|\leqq r$, $f(z)\neq0$ on $|z|=r$. D. C. Spencer [Amer. J. Math. **65**, 147–160 (1943); these Rev. **4**, 137] has proved the identity

$$(1)\quad r\frac{d}{dr}\int_{-\pi}^{\pi}G(|f(re^{i\phi})|)\,d\phi=\int_{-\pi}^{\pi}\int_0^r g'(|f|)\,|f|^{-1}|f'|^2\rho\,d\rho\,d\phi+2\pi g(0)n(r,0),$$

where $n(r,w)$ denotes the number of zeros of $f(z)-w$ in $|z|<r$. The author, assuming $g(R)$ continuous for $R\geqq0$, $g'(R)$ continuous for $R>0$, gives a new proof of the identity (1) by an application of Green's formula to the function $G(|f(z)|)$, and extends the result to meromorphic functions $f(z)$. An analogous result is obtained for harmonic functions when $G(R)$ is of the form $H(HR^t)$, $H''(t)$ continuous for $t\geqq1$.

M. S. Robertson (New Brunswick, N. J.).

Flett, T. M. A note on conformal mapping. J. London Math. Soc. **29**, 118–121 (1954).

Another proof is given of the theorem that the function $w=f(z)$ mapping a region G, bounded by a Jordan curve, conformally on the unit circle Ω: $|w|<1$ is continuous and 1-1 in $\bar{G}$ and f^{-1} is continuous in $\bar{\Omega}$. *G. Springer.*

Zakarin, A. On a method of successive approximations in conformal mapping. Izvestiya Akad. Nauk Kazah. SSR **1951**, no. 62, Ser. Mat. Meh. **5**, 104–118 (1951). (Russian)

A method of successive approximations is used to construct an analytic function which maps the unit circle $|z|<1$ conformally on a simply connected nearly circular region. The successive approximations are based on the familiar integral representation of the sum of a conjugate trigonometric series. The author's method is compared, in some special cases, with that of L. V. Kantorovich [Conformal mapping of simply and multiply connected regions, ONTI, Leningrad-Moscow, 1937, pp. 5–17]. *W. Seidel.*

Paatero, V. Über die Verzerrung bei der Abbildung mehrblättriger Gebiete von beschränkter Randdrehung. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **147**, 7 pp. (1953).

The present paper is a continuation of a previous one [same Ann. no. **128** (1952); these Rev. **14**, 861]. The author finds a function $\zeta(z)$ which maps the unit circle onto a finitely-sheeted domain of bounded boundary-rotation. He derives for $|\zeta'(z)|$ precise estimations together with explicit extremal functions, and further an estimation for $|\zeta''(z)|$, which is however precise only under an additional condition on $\zeta(z)$. *Y. Komatu* (Tokyo).

Trohimčuk, Yu. Yu. On removable boundary sets. Ukrain. Mat. Žurnal **4**, 312–322 (1952). (Russian)

The paper, which restricts itself to plane sets, consists of comments on two criteria for removability due to L. Sario [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. **50** (1948); these Rev. **10**, 365]. Of the two criteria concerned, one includes the other as a special case and the author shows this to be a proper inclusion. He shows further, inter alia, that a set satisfying the first criterion meets rectifiable curves in sets of zero length and that a set satisfying the second criterion has projections of zero length on the axes.

L. C. Young (Madison, Wis.).

Lotz, Wilhelm. Zur Streckenkomplexdarstellung Riemannscher Flächen. Mitt. Math. Sem. Giessen no. **39**, 25 pp. (1951).

This thesis constitutes an extension of the work of Blanc [Comment. Math. Helv. **9**, 193–216, 335–368 (1937)] and Drape [Deutsche Math. **1**, 805–824 (1936)] on the Riemann surfaces studied by R. Nevanlinna which have only a finite number of ramification points, all logarithmic. In the classification theory of Drape it was assumed that there was at most one ramification point of the surface over any point

of the complex sphere. In the present paper this assumption is dropped and a study is made of the classification of such surfaces. Application is made to special surfaces studied by Wagner [J. Reine Angew. Math. **175**, 6–49 (1936)].

M. Heins (Providence, R. I.).

Martinelli, Enzo. Sulle estensioni della formula integrale di Cauchy alle funzioni analitiche di più variabili complesse. Ann. Mat. Pura Appl. (4) **34**, 277–347 (1953).

L'auteur démontre, pour les fonctions holomorphes de n variables complexes, des résultats généralisant la formule intégrale de Cauchy $2\pi i N f(\zeta)=\int_{\Gamma_1} f(z)(z-\zeta)^{-1}dz$ où $f(z)$ est une fonction holomorphe d'une variable complexe, Γ_1 un 1-cycle ne passant pas par ζ, homologue à 0 dans le domaine d'holomorphie de f et N le coefficient d'enlacement de Γ_1 et du point ζ. Ces résultats ont été annoncés dans une Note [Pont. Acad. Sci. Acta **9**, 235–250 (1945); ces Rev. **10**, 111] et sont du type suivant: étant donnée une fonction holomorphe f sur un ouvert R_{2n} de l'espace numérique complexe $S_{2n}(z_1, \cdots, z_n)$ contenant le point $O(\zeta_1, \cdots, \zeta_n)$, il existe $\binom{n}{l}$ formules intégrales correspondant aux combinaisons $k_1, \cdots, k_l$ des entiers $1, \cdots, n$:

$$N_{k_1\cdots k_l} f(\zeta_1, \cdots, \zeta_n)=\int_{\Gamma_{n+l}} f(z_1, \cdots, z_n)\varphi_{k_1\cdots k_l}.$$

L'expression $\varphi_{k_1\cdots k_l}$ est une forme différentielle fermée, de type (n, l) (forme noyau), dont les coefficients sont les quotients de fonctions rationnelles des quantités $(z_j-\zeta_j)\overline{(z_j-\zeta_j)}$, par les produits de $(n-l)$ facteurs $(z_i-\zeta_i)$ différents; si $T_{2n-2l-2}$ désigne la réunion des $\binom{n}{l+1}$ sous-espaces linéaires de S_{2n} définis par $z_{\alpha_1}=\zeta_{\alpha_1}, \cdots, z_{\alpha_{l+1}}=\zeta_{\alpha_{l+1}}$, où $\alpha_1, \cdots, \alpha_{l+1}$ est une combinaison des entiers $1, \cdots, n$, les coefficients de $\varphi_{k_1\cdots k_l}$ sont des fonctions analytiques (réelles) sur $S_{2n}-T_{2n-2l-2}$, possédant des singularités polaires sur $T_{2n-2l-2}$. On désigne par Γ_{n+l} un cycle singulier différentiable satisfaisant aux conditions: $\Gamma_{n+l}\subset R_{2n}-T_{2n-2l-2}$; Γ_{n+l} homologue à zéro dans $R_{2n}-T_{2n-2l-2}+O$ (pour l'homologie à coefficients rationnels). $N_{k_1\cdots k_l}$ désigne le coefficient d'enlacement de Γ_{n+l} avec un $(n-l-1)$-cycle de $T_{2n-2l-2}$ qui dépend du choix du noyau. Les formules ont été déjà démontrées, dans le cas $l=0$, par l'auteur [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **25**, 33–36 (1937)] et par B. Segre [Atti 1° Congresso Un. Mat. Ital., Firenze, 1937, Zanichelli, Bologna, 1938, pp. 174–180] et dans le cas $l=n-1$, par l'auteur [Accad. Ital. Mem. Cl. Sci. Fis. Mat. Nat. **9**, 269–283 (1938)]. La technique de la démonstration fait intervenir le théorème de dualité d'Alexander pour définir "l'état d'enlacement" de Γ_{n+l} et de $T_{2n-2l-2}$ et le théorème de Poincaré: toute forme différentielle fermée est, localement, une différentielle. Le noyau est construit par un procédé de récurrence, de manière à obtenir la forme la plus simple généralisant les expressions déjà connues pour $l=0$ et $l=n-1$.

P. Dolbeault (Paris).

***Bers, Lipman. Partial differential equations and pseudo-analytic functions on Riemann surfaces.** Contributions to the theory of Riemann surfaces, pp. 157–165. Annals of Mathematics Studies, no. 30. Princeton University Press, Princeton, N. J., 1953. $4.00.

The theory of pseudo-analytic functions developed by the author [Proc. Nat. Acad. Sci. U. S. A. **36**, 130–136 (1950); **37**, 42–47 (1951); Theory of pseudo-analytic functions, Inst. Math. Mech., New York Univ., 1953; these Rev. **12**, 173; **13**, 352; **15**, 211], is extended to functions defined on Riemann surfaces. A generating pair (F, G) on a Riemann surface S is a pair of functions $F(p)$, $G(p)$, $p\in S$, such that (i) $F\not\equiv 0$, $G\not\equiv 0$ and the ratio G/F has a positive imaginary part, (ii) F and G considered as functions of local parameters are sufficiently smooth. A function $w(p)$, $p\in D\subset S$ is called (F, G)-pseudo-analytic if it is an (F, G)-pseudo-analytic function (in the sense defined in the references given above) of a local parameter. A differential dw defined in $D\subset S$ is called (F, G)-pseudo-analytic if in the neighborhood of every point of D one has, in terms of a local parameter, $dw=\dot{W}dz$, $\dot{W}$ being an (F, G) derivative. A function (differential) defined over the whole surface S and (F, G)-pseudo-analytic except for poles is called an (F, G)-function ((F, G)-differential) on S. To each generating pair (F, G) is associated a pair (F^*, G^*) by the relations $F^*=2G/(F\bar{G}-\bar{F}G)$, $G^*=2\bar{F}/(F\bar{G}-\bar{F}G)$. For $F=1$, $G=i$ one obtains the classical theory of meromorphic functions and differentials.

Let S be a closed Riemann surface of genus g, C a closed curve on S and dw a differential on S. The number $\operatorname{Re}\oint_C F^*dw-i\operatorname{Re}\oint_C G^*dw$ is called the (F, G) period of dw over C. The theorem is stated: Let $p_1, \cdots, p_k$, $k\geqq 0$, be distinct points on S and let dw_j, $j=1, 2, \cdots, k$, be an (F, G) pseudo-analytic meromorphic differential in a neighborhood of p_j and having at p_j a pole with residue σ_j. If $\sigma_1+\cdots+\sigma_k=0$, then there exists on S exactly one (F, G)-differential dw such that $dw-dw_j$ is regular at p_j and all periods of dw on non-bounding curves are purely imaginary.

The following generalization of the Riemann-Roch theorem is given. Let d be a divisor on S of order m, A the number of linearly independent (F^*, G^*)-functions which are multiples of $1/d$, B the number of linearly independent (F, G) differentials which are multiples of d. Then $A-B=2(m+1-g)$. Proofs of these theorems are omitted. The cases of surfaces with genus 0, 1 are discussed, an application to multiple-valued solutions of

$$\varphi_{zz}+\varphi_{ww}+\alpha(x, y)\varphi_x+\beta(x, y)\varphi_y=0$$

is given, and in the final section some unsolved problems are stated.

M. H. Protter (Berkeley, Calif.).

Theory of Series

Hara, Hisao. On the Cauchy's product series theorem on Euler's summability. Kōdai Math. Sem. Rep. **1953**, 91–92 (1953).

If two series $\sum a_n$ and $\sum b_n$ are absolutely evaluable by the Euler-Knopp method (E, p) to A and B respectively, then their Cauchy product series $\sum c_n$ is absolutely evaluable (E, p) to AB.

R. P. Agnew (Ithaca, N. Y.).

Sunouchi, Gen-ichirô. Tauberian theorems for Riemann summability. Tôhoku Math. J. (2) **5**, 34–42 (1953).

The author treats the two different Riemann methods of order 1 for evaluation of series $a_1+a_2+\cdots$ with partial sums $s_1, s_2, \cdots$. If $\sum_{k=1}^{n}s_k=O(n^\alpha)$ and $\sum_{k=n}^{\infty}|a_k/k|=O(n^{-\alpha})$ when $0<\alpha<1$, then $\sum a_n$ is evaluable R_1 to 0, that is, $\sum(s_k/k)\sin kt\to 0$ as $t\to 0+$. Under the same hypotheses, $\sum a_n$ is evaluable $(R, 1)$ to 0, that is, $\sum(a_k\sin kt)/kt\to 0$ as $t\to 0+$. The proofs of the above theorems lead the author to a proof of the following theorem. If $\sum_{k=1}^{n}ka_k=O(n^\alpha)$ and $\sum_{k=n}^{\infty}|\Delta a_k|=O(n^{-\alpha})$ when $0<\alpha<1$, then the trigonometric series $\sum a_k\sin kt$ converges uniformly over $0\leqq t\leqq\pi$.

R. P. Agnew (Ithaca, N. Y.).

Tatchell, J. B. A theorem on absolute Riesz summability. J. London Math. Soc. **29**, 49–59 (1954).

Let $k \geqq 0$ and let $0<\lambda_1<\lambda_2<\cdots$ and $\lambda_n \to \infty$. Hardy and Riesz [The general theory of Dirichlet series, Cambridge, 1915] have shown that if $\sum c_n$ is evaluable by the Riesz method (R, λ_n, k), then $\sum c_n \lambda_n^{-k}$ is evaluable by the method (R, e^{λ_n}, k). The author proves the corresponding theorem for absolute evaluability by Riesz methods. The theorem was proved by R. Mohanty [same J. **25**, 67–72 (1950); these Rev. **11**, 592] for the special case in which $r=1$. The author's result and further arguments imply that, for each Dirichlet series $\sum a_n e^{-\lambda_n s}$, the abscissae of evaluability (whether finite or infinite) by the methods $|R, \lambda_n, k|$ and $|R, e^{\lambda_n}, k|$ are the same. *R. P. Agnew.*

Peyser, Gideon. Sur les théorèmes d'Abel et de Tauber pour des séries entières à n variables. C. R. Acad. Sci. Paris **237**, 1135–1137 (1953).

Verfasser vergleicht die (durch Umordnung in eine gewisse Einfachreihe erklärte) Konvergenz der Reihe $\sum A_{p_1,\cdots,p_n}$ mit dem Grenzverhalten von $\sum A_{p_1,\cdots,p_n} z_1^{p_1}\cdots z_n^{p_n}$, wenn die z_i unter gewissen Einschränkungen gegen 1 streben. Er verallgemeinert unter anderem den Umkehrsatz für das Abelverfahren mit der Bedingung $O(1/k)$. Durch Beispiele belegt er die Notwendigkeit zweier Voraussetzungen über die Wege $z_i \to 1$. Mehrere Autoren erhielten ähnliche Resultate bei Verwendung anderer Konvergenzbegriffe, siehe zum Beispiel K. Knopp [Math. Z. **45**, 573–589 (1939); diese Rev. **1**, 51], I. I. Ogieveckiĭ [Doklady Akad. Nauk SSSR (N.S.) **58**, 1897–1900 (1947); diese Rev. **9**, 278], V. G. Čelidze [ibid. (N.S.) **60**, 553–554 (1948); diese Rev. **9**, 507]. *K. Zeller* (Philadelphia, Pa.).

Agnew, Ralph P. Mercer's summability theorem. J. London Math. Soc. **29**, 123–125 (1954).

The author points out that some theorems of Love [same J. **27**, 413–428 (1952); these Rev. **14**, 159] follow from known results [Agnew, Tôhoku Math. J. **35**, 244–252 (1932); Rado, Quart. J. Math., Oxford Ser. **9**, 274–282 (1938)]. *G. G. Lorentz* (Detroit, Mich.).

Jurkat, Wolfgang, und Peyerimhoff, Alexander. Summierbarkeitsfaktoren. Math. Z. **58**, 186–203 (1953).

In extension of theorems of Hadamard, Hardy, Bromwich, Bohr, Fekete, Kojima, Schur and Andersen, it has been proved that necessary and sufficient conditions for $\sum a_n \epsilon_n$ to be summable (C, β) whenever

$$a_0+\cdots+a_n=o(n^p)(C, \alpha),$$

where $p \geqq 0$, $0 \leqq \beta \leqq \alpha$, are that

(i) $\epsilon_n = O(n^{-p+\beta-\alpha})$, (ii) $\sum n^{\alpha+p}|\Delta^{\alpha+1}\epsilon_n| < \infty$

[Bosanquet, Proc. London Math. Soc. (2) **50**, 482–496 (1949); these Rev. **10**, 368; see also Knopp, J. Reine Angew. Math. **187**, 70–74 (1949) (for the case α an integer, $p=0$); these Rev. **11**, 512; for references to earlier results see Hardy, Divergent series, Oxford, 1949, p. 146; these Rev. **11**, 25].

If

$$a_{n\nu}=\binom{n-\nu+\alpha-1}{n-\nu}\Big/\binom{n+\alpha}{n},$$

$$b_{n\nu}=\binom{n-\nu+\beta-1}{n-\nu}\Big/\binom{n+\beta}{n}$$

and $\eta_n = o(n^p)$, the conditions may be written:

(i)′ $\epsilon_n = O\left(\frac{1}{\eta_n}\frac{a_{nn}}{b_{nn}}\right)$, (ii)′ $\sum|\eta_n \alpha_n| < \infty$,

where α_n satisfies $\Delta\epsilon_n = \sum_{\mu=n}^{\infty} a_{\mu n}\alpha_\mu$.

If $0 \leqq \beta \leqq \alpha \leqq 1$, the following inequalities hold:

$$M(A): \quad \left|\sum_{\nu=0}^{n} a_{m\nu}s_\nu\right| \leqq K\left|\sum_{\nu=0}^{n'} a_{m'\nu}s_\nu\right|$$

for $0 \leqq n \leqq m$ and suitable n', with $0 \leqq n' \leqq n$ [first stated by Jacob, Proc. London Math. Soc. (2) **26**, 470–492 (1927) and given independently by the reviewer, J. London Math. Soc. **16**, 146–148 (1941) and loc. cit. supra; these Rev. **3**, 144; compare Riesz's mean value theorem, Hardy and Riesz, The general theory of Dirichlet's series, Cambridge, 1915, reprinted, 1952, Lemma 7];

$$V(A, B): \quad \left|a_{m'n}\sum_{\nu=0}^{n} b_{m\nu}s_\nu\right| \leqq K\left|b_{m'n}\sum_{\nu=0}^{n''} a_{n'\nu}s_\nu\right|$$

for $0 \leqq n \leqq m' \leqq m$ and suitable n', n'', with $n'' \leqq n'$, $n'' \leqq n$, $n' \leqq m$.

In continuation of their earlier papers [Peyerimhoff, Math. Z. **55**, 23–54 (1951); Jurkat and Peyerimhoff, ibid. **55**, 92–108 (1951); **56**, 152–178 (1952); Jurkat, ibid. **57**, 353–394 (1953); these Rev. **13**, 933, 934; **14**, 158, 866] the authors show, for general triangular matrices $A=(a_{n\nu})$, $B=(b_{n\nu})$ satisfying $M(A)$, $V(A, B)$ and certain other conditions, that $\sum a_n\epsilon_n$ is summable (B) whenever $a_0+\cdots+a_n=o(\eta_n)$ (A), where η_n increases, if and only if (i)′ and (ii)′ hold. The proof is independent of functional analysis except (apparently) at one point, where the authors cite earlier proofs of their Lemma 4. But an elementary proof of the convergence of $\sum s_n\Delta\epsilon_n$ may also be given [compare arguments in the first paper of the reviewer cited above, p. 492, or the first joint paper of the authors, p. 102].

Applications are given to the summability methods of weighted arithmetic means, Nörlund means and Riesz means. *L. S. Bosanquet* (London).

Meyer-König, W., und Zeller, K. Inäquivalenzsätze bei Limitierungsverfahren. Math. Z. **59**, 200–205 (1953).

The authors give a direct proof (independent of Baire's category theorem) of the following result of Zeller [Math. Z. **53**, 463–487 (1951); these Rev. **12**, 604]. Let A and A_i $(i=1, 2, \cdots)$ be infinite matrices. If for each i there is an A summable sequence which is not A_i summable, then there is an A summable sequence which is not A_i summable for $i=1, 2, \cdots$. [This is an "inequivalence theorem"; a first theorem of this type, for almost convergence, has been given by the reviewer, Leningrad. Gos. Univ. Učenye Zapiski **83**, Ser. Mat. Nauk **12**, 30–41 (1941); these Rev. **7**, 517.] It follows that such methods as E_n and $\bigcup_{k=1}^{\infty} E_k$ (where E_k is the Euler-Knopp method of order k) are different from any matrix method. *G. G. Lorentz.*

Mo, Yeh. On d_n-monotone sequence. Acad. Sinica Science Record **5**, 51–58 (1952). (Chinese summary)

Let $0<d_1<d_2<\cdots$ and $d_n \to \infty$. A sequence $a_1, a_2, \cdots$ and a series $\sum a_n$ are said to be d_n-monotone if $a_n>0$ for each n, and for some positive constant c, $a_{n+1} \leqq a_n(1+c/d_n)$, for each sufficiently great n. Let $\sum a_n$ be d_n-monotone. Let p be an integer greater than 1 and let $k_n=[n/p]$. Let $d_n=O(n)$ and let $d_n/d_{k_n}=O(1)$. These conditions imply that if $\sum a_n$ converges, then $d_n a_n \to 0$ as $n \to \infty$. If d_n satisfies all of the above conditions, if $\sum b_n$ is d_n-monotone, if $\sum a_n$ has

bounded partial sums, and if $\sum b_n/d_n$ converges, then $\sum a_n b_n$ converges. *R. P. Agnew* (Ithaca, N. Y.).

Ščeglov, M. P. On a generalization of a theorem of Hardy-Littlewood. Ukrain. Mat. Žurnal **5**, 299–303 (1953). (Russian)

Let us consider the set P of all non-negative sequences $s_0, s_1, s_2, \cdots$ and let

$$\phi(u)=u^{-1}\sum_{n=0}^{\infty}s_n e^{-n/u}, \quad \sigma_n=(n+1)^{-1}\sum_{\nu=0}^{n}s_\nu$$

be their Abel and $(C, 1)$ means. Let

$$\limsup_{n\to\infty}\sigma_n=D, \quad \limsup_{u\to\infty}\phi(u)=D'.$$

It is a familiar fact that D and D' are either both finite or both infinite, and the classical proof of Hardy and Littlewood [Proc. London Math. Soc. (2) **13**, 174–191 (1914)] shows that $D\leqq eD'$. Assuming that both D and D' are finite, the author proves that a) $\inf_P (D-D')=0$; b) $\sup_P (D-D')=+\infty$; c) $\inf_P D/D'=1$; d) $\sup_P D/D'=e$; e) $\sup_P (D+\alpha)/(D'+\alpha)=e$, for any finite positive α. Also, 1) if $\limsup s_n=D$, then $D'=D$; 2) there exist $\{s_n\}\in P$ such that $D=D'<\limsup s_n$. *A. Zygmund.*

Cooper, R. The relative growth of some rapidly increasing sequences. J. London Math. Soc. **29**, 59–62 (1954).

Let $\log\log f_{i,n+1}=\log\alpha_i+\sigma_i\log f_{i,n}$, $i=1, 2$ and $n=0, 1, 2, \cdots$, where $\alpha_i>0$, $\sigma_i>0$ and $f_{i,0}$ is sufficiently large to guarantee $f_{i,n}\uparrow$ as $n\to\infty$. Let $n(m)$ be defined by $f_{1,n(m)}<f_{2,m}\leqq f_{1,n(m)+1}$. Then for m sufficiently large $n(m)-m$ is constant. *P. Civin* (Princeton, N. J.).

Rajagopal, C. T. Note on a class of Tauberian series. Duke Math. J. **20**, 617–620 (1953).

Let

$$\psi(u)\geqq 0, \quad \int_0^\infty\psi(u)du=1, \quad \phi(u)=\int_u^\infty\psi(x)dx.$$

Let $0=\lambda_0<\lambda_1<\cdots$, let $\lambda_{n+1}/\lambda_n\to 1$, let $\lambda_n\to\infty$, let $a_1+a_2+\cdots$ be a series of complex terms, and let $A(u)=a_1+a_2+\cdots+a_n$, when $\lambda_n\leqq u<\lambda_{n+1}$. Let $\Phi(s)=\int_0^\infty\varphi(us)dA(u)$. Several properties of the transform $\Phi(s)$ are determined for the case in which $\sum a_n$ satisfies the Tauberian condition

$$\lim_{n\to\infty}\lambda_n|a_n|/(\lambda_n-\lambda_{n-1})=h>0,$$

and the partial sums $A(\lambda_n)$ lie on and progress steadily in a positive direction along a closed rectifiable curve C as n increases. The limit points, as $s\to 0+$, of $\Phi(s)$ constitute a closed rectifiable curve C_h about which some precise information is given. *R. P. Agnew* (Ithaca, N. Y.).

Fourier Series and Generalizations, Integral Transforms

***Natanszon, I. P. Konstruktiv függvénytan.** [Constructive theory of functions.] Akadémiai Kiadó, Budapest, 1952. 517 pp. 80 Ft.

Translation by A. Rényi of Natanson's Konstruktivnaya teoriya funkciĭ [Gostehizdat, Moscow-Leningrad, 1949; these Rev. 11, 591].

Pyateckiĭ-Šapiro, I. I. Fractional parts and some questions of the theory of trigonometric series. Uspehi Matem. Nauk (N.S.) **8**, no. 3(55), 167–170 (1953). (Russian)

The author gives a summary, without proofs, of his results published elsewhere [see, in particular, Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 47–52 (1951); Doklady Akad. Nauk SSSR (N.S.) **85**, 497–500 (1952); these Rev. **13**, 213; **14**, 161]. Another paper whose results he quotes and which has appeared in the Moskov. Gos. Univ. Učenye Zapiski **155**, Matematika **5** (1952) is still not available for review. *A. Zygmund* (Cambridge, England).

Pati, T. On the absolute Riesz summability of Fourier series and its conjugate series. Bull. Calcutta Math. Soc. **44**, 155–168 (1952).

Let $f(t)$ be an integrable function with period 2π and let

$$\varphi(t)=f(x+t)+f(x-t)-2f(x), \quad \psi(t)=f(x+t)-f(x-t),$$

$$\varphi_2(t)=\frac{2}{t^2}\int_0^t(t-u)\varphi(u)du, \quad \psi_2(t)=\frac{2}{t^2}\int_0^t(t-u)\psi(u)du.$$

Then the author proves that (1) if $\varphi_2(t)\log(\pi/t)$ is of bounded variation in $(0, \pi)$, then the Fourier series of $f(t)$, at $t=x$, is summable $|R, \exp(\log\omega)^{3/2}, 3|$, and (2) if $\psi_2(t)\log(\pi/t)$ is of bounded variation in $(0, \pi)$ and $|\psi_2(t)|/t$ is integrable in $(0, \pi)$, then the conjugate Fourier series of $f(t)$, at $t=x$, is also summable by the same method. The author proves similar theorems for the derived Fourier series and its conjugate series. These are closely related to the work of R. Mohanty [Proc. London Math. Soc. (2) **51**, 186–196 (1949); these Rev. **11**, 99]. *S. Izumi.*

Shapiro, Victor L. An extension of results in the uniqueness theory of double trigonometric series. Duke Math. J. **20**, 359–365 (1953).

Soit $\sum_{m,n}a_{m,n}\exp(i(mx+ny))$ une série trigonométrique double. Appelons $S(R; x, y)$ la somme $\sum_{m^2+n^2\leqq R^2}$, et $\sigma(R; x, y)$ la moyenne de Cesàro $(2/R^2)\int_0^R S(u; x, y)udu$. La série sera dite convergente en x, y, si $S(R; x, y)$ a une limite pour $R\to\infty$, et $(C, 1)$-convergente en x, y, si $\sigma(R; x, y)$ a une limite pour $R\to\infty$. L'auteur améliore des résultats de Cheng [Ann. of Math. (2) **52**, 403–416 (1950); ces Rev. **12**, 174]; voici le résultat essentiel: Si $\lim_{m,n\to\infty}a_{m,n}=0$, si la série $\sum_{m,n}(a_{m,n}/(m^2+n^2))\exp(i(mx+ny))$ converge uniformément, si la série donnée converge $(C, 1)$ presque partout vers une function f essentiellement bornée, si enfin les $|\sigma(R; x, y)|$ de la série donnée ont une limite supérieure finie quasi-partout (=sauf sur un ensemble de capacité nulle), alors les $a_{m,n}$ sont les coefficients de Fourier de f.

La démonstration utilise le laplacien généralisé défini par des moyennes circulaires et un critère d'harmonicité correspondant. *L. Schwartz* (Paris).

Rabson, Gustave. Summability of Fourier series on the quaternions of norm one. Trans. Amer. Math. Soc. **75**, 287–303 (1953).

Let G be a compact, second-countable group, χ^α the character of the αth irreducible representation, of degree r_α. The character series of a function $f\in L^1(G)$ is defined by $\sum_\alpha r_\alpha(f*\chi^\alpha)$. After some general remarks, the author considers the group Q of quaternions of norm one, and proves the analogues of the Fejér and Fejér-Lebesgue theorems, but with $(C, 2)$-summation replacing $(C, 1)$. There follows the Kolmogoroff theorem, that for $f\in L^2(Q)$, $n_{k+1}/n_k>C>1$, the subsequence $\{S_{n_k}\}$ of the partial sums of the character series of f converges to f almost everywhere. Abel summa-

bility at a point of continuity is proved independently. Finally, it is shown that the localization theorem fails on Q. [There are several misprints, the only serious one being in Th. 12.1, where the condition should be "For $f \in L^1 \cdots$". The remark at the beginning of §13 should be ignored.]

N. J. Fine (Princeton, N. J.).

Sabbioni, C. Sopra un esempio di funzione quasi periodica. Boll. Un. Mat. Ital. (3) **8**, 301–303 (1953).

Exemple très simple d'une fonction presque-périodique à dérivée presque-partout bornée, mais non presque-périodique au sens de Stepanoff. *J. Favard* (Paris).

Hartman, S., et Ryll-Nardzewski, C. Théorèmes abstraits de Kronecker et les fonctions presque périodiques. Studia Math. **13**, 296–310 (1953).

This paper presents a new proof and applications in various directions of an approximation theorem of the reviewers [Ann. of Math. (2) **52**, 557–567 (1950); these Rev. **12**, 801]. A set of characters $\{\chi_\lambda\}$ of a group is said to be correlated with a set $\{\alpha_\lambda\}$ of complex numbers of absolute value 1 if $\chi_{\lambda_1}^{k_1} \cdots \chi_{\lambda_n}^{k_n} = 1$ implies that $\alpha_{\lambda_1}^{k_1} \cdots \alpha_{\lambda_n}^{k_n} = 1$, for all finite sets of indices $\lambda_1, \cdots, \lambda_n$ and integers $k_1, \cdots, k_n$. The authors formulate the reviewers' theorem as follows: if the characters $\chi_1, \cdots, \chi_n$ of a group G are correlated with the numbers $\alpha_1, \cdots, \alpha_n$, then for every $\epsilon > 0$ there exists $t \in G$ such that $|\chi_j(t) - \alpha_j| < \epsilon$ $(j = 1, 2, \cdots, n)$. The proof, which is elementary in character, depends upon the abstract theory of almost periodic functions. The following theorems are proved. If f is an almost periodic function on an Abelian group G with Fourier series $\sum a_n \chi_n$, and if the characters χ_n are correlated with the numbers $\operatorname{sgn} \overline{a_n}$, then $\sum |a_n| < \infty$ and $\sum |a_n| = \sup_{t \in G} |f(t)|$. This generalization of a well-known theorem of H. Bohr has also been proved by van Kampen for arbitrary almost periodic functions [Ann. of Math. **37**, 78–91 (1936)]. If f is an almost periodic function on an Agelian group G with absolutely convergent Fourier series and if $\inf_{t \in G} |f(t)| > 0$, then f^{-1} also has absolutely convergent Fourier series. The authors' proof is a repetition of Gel'fand's proof of Wiener's original theorem (G = the circle group) and hence makes use of the axiom of choice.

E. Hewitt and *H. S. Zuckerman* (Seattle, Wash.).

Beurling, Arne, and Helson, Henry. Fourier-Stieltjes transforms with bounded powers. Math. Scand. **1**, 120–126 (1953).

The set A of all functions $f(t) = \int_{-\infty}^{\infty} e^{ixt} d\sigma(x)$, σ being a complex function of finite variation, is a Banach algebra under the ordinary algebraic operations and with $\|f\| = \int_{-\infty}^{\infty} |d\sigma(x)|$. Theorem: if $f \in A$ and $\{\|f^n\|\}_{n=-\infty}^{\infty}$ is a bounded set of numbers, then $f(t) = e^{i(at+b)}$, where a and b are real constants. The proof is strictly constructive. The corresponding theorem is stated for a certain class of groups. If G is a locally compact Abelian connected group, if f is a Fourier-Stieltjes transform on G all of whose powers are bounded in spectral norm, then f is a continuous character of G [reviewer's note: times a complex constant of absolute value 1]. The following corollary is also stated: if G and H are locally compact Abelian groups, if G or H is connected, and if $L_1(G)$ and $L_1(H)$ are algebraically isomorphic, then G and H are topologically isomorphic. *E. Hewitt.*

Bosanquet, L. S. The summability of Laplace-Stieltjes integrals. Proc. London Math. Soc. (3) **3**, 267–304 (1953).

This is a systematic study of evaluability, by Cesàro methods of real orders, of Laplace-Stieltjes integrals $\int_0^\infty e^{-su} dA(u)$ where $s = \sigma + i\tau$ and of related Mellin-Stieltjes integrals. About 40 theorems and lemmas deal with questions involving evaluability and absolute evaluability of the integrals for given values of s and with questions involving half-planes of evaluability. The origin of the paper lies in the observation that if $k \geq 1$ then the abscissa of evaluability (C, k) of the above Laplace-Stieltjes integral is, when non-negative, the same as the abscissa of evaluability $(C, k-1)$ of the integral $\int_0^\infty e^{-su} A(u) du$. Corollaries of the theorems fill gaps in the literature concerning abscissas of evaluability of Dirichlet series. An appendix gives general theorems on function-to-function kernel transformations.

Errata: Page 295, line 9 above bottom, replace "logarithmic means" by "means of the second kind". Page 299, line 15 above bottom, replace "Laplace-Stieltjes" by "Mellin-Stieltjes". *R. P. Agnew* (Ithaca, N. Y.).

Isaacs, G. L. On a limitation theorem for Laplace integrals. J. London Math. Soc. **28**, 329–335 (1953).

Let $A(u)$ be of bounded variation in every finite interval and let:

$$A_\kappa(u) = \frac{1}{\Gamma(\kappa)} \int_0^u (u-t)^{\kappa-1} A(t) dt.$$

It was proved by Bosanquet [see the paper reviewed above] that if the Laplace transform $\int_0^\infty \exp(-su) dA(u)$ is absolutely Cesàro summable of integral order $\kappa \geq 0$ for some $s_0 = \sigma_0 + it_0$ with $\sigma_0 > 0$, then: $\exp(-us) u^{-\kappa} A_\kappa(u) \to 0$ as $u \to \infty$ and is of bounded variation over $[1, \infty)$ for every s with $\sigma = \sigma_0$. The author generalizes the theorem to fractional κ. *S. Agmon* (Jerusalem).

Voelker, Dietrich. On convergence of the Laplace integral. An Soc. Ci. Argentina **155**, 119–133 (1953). (Spanish. German summary)

The author demonstrates several well known theorems concerning the regions of uniform convergence of a Laplace transform. *I. I. Hirschman, Jr.* (St. Louis, Mo.).

Lions, J. L. Supports dans la transformation de Laplace. J. Analyse Math. **2**, 369–380 (1953).

Soit $X^n = R^n$, Ξ^n son dual, $\Pi^n = \Xi^n + i\Xi^n$ l'espace complexe associé. Γ est un ensemble convexe de Ξ^n d'intérieur non vide. $\mathcal{S}'(\Gamma)$ est l'espace des distributions T sur X^n telles que $\exp(-\langle \xi, x \rangle) T_x$ soit tempérée sur X^n pour tout $\xi \in \Gamma$. Si $T \in \mathcal{S}'(\Gamma)$, sa transformée de Laplace $F(p)$, $p = \xi + i\eta$, est une fonction holomorphe sur $\mathring{\Gamma} + i\Xi^n$ [voir L. Schwartz, Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 196–206 (1952); ces Rev. **14**, 639]. $K(T)$ est l'enveloppe convexe fermée du support de T; son indicatrice est la fonction $k(\lambda) = \inf_{x \in K(T)} \langle \lambda, x \rangle$, définie pour les vecteurs unitaires λ de Ξ^n. L'auteur a donné un théorème reliant $K(T)$ à la croissance de F dans le domaine complexe (article du reviewer cité ci-dessus). Il relie ici $K(T)$ à la croissance de F dans le domaine réel. Si $n = 1$, et si $F(p) = O(\exp(\epsilon |p|^2))$ pour $|p| \to \infty$ et tout $\epsilon > 0$, et si A est la borne supérieure des nombres réels b tels que $\exp(b\xi) F(\xi) = O(1)$ pour $\xi \to \infty$, alors $k(+1) = A$. La démonstration s'appuie sur le théorème de Phragmen-Lindelöf. Pour n quelconque, si $F(p + \lambda\zeta) = O(\exp(\epsilon |\zeta|^2))$ pour $|\zeta| \to \infty$ $(\Re \zeta \geq 0)$ pour tout $\epsilon > 0$, et si $A(p; \lambda)$ est la borne supérieure de l'ensemble des nombres réels b tels que $\exp(bt) F(p + \lambda t) = O(1)$ pour $t \to \infty$, alors $k(\lambda) \leqq A(p; \lambda)$, et, pour λ fixé, $k(\lambda) = A(p; \lambda)$ sauf lorsque p varie dans un ensemble exceptionnel de droites complexes de direction λ qui est "presque dénombrable" (c.a.d. coupe toute droite com-

plexe non parallèle suivant un ensemble dénombrable). On en déduit de nouvelles démonstrations du théorème de Paley-Wiener sur la transformée de Fourier d'une distribution à support compact, et d'un théorème antérieur de l'auteur, $K(S*T)=K(S)+K(T)$ [C. R. Acad. Sci. Paris 232, 1530–1532, 1622–1624 (1951); ces Rev. 13, 231]. Il y avait une erreur dans ces notes, "dénombrable" (en bas page 1531) doit être remplacé par "presque dénombrable"; l'erreur est rectifiée ici). *L. Schwartz* (Paris).

Włodarski, L. Une remarque sur une classe de fonctions exponentielles du calcul opérationnel. Studia Math. 13, 188–189 (1953).

L'auteur démontre que la fonction $F(z)=\exp(-\lambda z^{\alpha})$, $\lambda>0$, $0<\alpha<1$, est transformée de Laplace d'une fonction $f(t)\geqq 0$ (on utilise le fait que $F(x)$ est complètement monotone pour x réel $\geqq 0$: $(-1)^k F^{(k)}(x)\geqq 0$). *L. Schwartz* (Paris).

Ward, E. E. The calculation of transients in dynamical systems. Proc. Cambridge Philos. Soc. 50, 49–59 (1954).

Let $F_1(s)$ denote the Laplace transform of a function of the type $F(t)=\sum_{n=1}^{m} A_n e^{b_n t}$ where $\Re(b_n)\leqq 0$. Then $F_1(s)$ can be written as a quotient of polynomials in s. The author presents a modification of methods due to Tricomi and others for approximating $F(t)$ without finding the zeros of the denominator of $F_1(s)$. The procedure involves a representation of the function $(s+b)F_1(s)$ in powers of $(s-b)/(s+b)$, where b denotes some positive constant, followed by a termwise inverse transformation that gives $F(t)$ as the product of an exponential function by a series of Laguerre polynomials. He discusses the problem of selecting the value of b so that the series will converge rapidly to $F(t)$; but he is unable to find a very good solution to that problem. Numerical examples are given. *R. V. Churchill.*

Söhngen, H. Algebraisierung der endlichen Hilbert-Transformation. Z. Angew. Math. Mech. 33, 280–281 (1953).

The author points out that the linear integral transformation

$$T\{F(x)\}=\int_{-1}^{1} K(x,s)F(x)dx=f(s),$$

where $K(x,s)=(1-x)^s(1+x)^{-s}$, with appropriate qualifications on the functions $F(x)$ and the complex parameter s, transforms the operation $H\{F(x)\}=\pi^{-1}\int_{-1}^{1}F(\alpha)(x-\alpha)^{-1}d\alpha$ into an algebraic operation on $f(s)$. The operator H is the basis of the finite Hilbert transformation. The inverse of T is represented as an integral transformation in the complex plane of s. He illustrates these results by solving a system of integral equations involving H. The unknowns in the system represent the vertical and horizontal components of the pressures at the base of a long dam as functions of the distance from the vertical face of the dam.

R. V. Churchill (Ann Arbor, Mich.).

Polynomials, Polynomial Approximations

Rostovcev, N. A. On an iterative solution of equations of odd degree with positive coefficients. Uspehi Matem. Nauk (N.S.) 7, no. 3(49), 135–138 (1952). (Russian)

Let the polynomial $F(z)=\sum A_j z^{m-j}$ be written in the form $F(z)=G(z^2)+zH(z^2)$. It is here shown that if (1) m is odd, (2) all $A_j>0$ and

$$\frac{A_1}{A_0}>\frac{A_3}{A_2}>\cdots>\frac{A_m}{A_{m-1}}, \tag{3}$$

and if $z_{k+1}=-G(z_k^2)/H(z_k^2)$, then $\lim z_k$ exists (convergence is monotone) and is the greatest or least root of $F(z)=0$ according as $z_0=0$ or $-A_1$. For the case $m=3$ this result was given by K. F. Teodorčik [Akad. Nauk SSSR. Žurnal Tehn. Fiz. 19, 231–234 (1949); these Rev. 12, 209].

R. Church (Monterey, Calif.).

Picone, Mauro. Maggiorazione di un polinomio di Legendre e delle derivate in un'ellisse a quello confocale. Boll. Un. Mat. Ital. (3) 8, 237–242 (1953).

All derivatives of Legendre's polynomial assume in an ellipse with foci at ± 1 their maximum modulus at the vertices of the ellipse. The very elementary proof is based on the generating function. *G. Szegő.*

Kanter, L. H. On a separation theorem for the zeros of the ultraspherical polynomials. Proc. Amer. Math. Soc. 4, 729–733 (1953).

Let $P_n^{(\lambda)}(x)=P_n(x,\lambda)=k_n(\lambda)x^n+\cdots$ denote the nth ultraspherical polynomial. The author studies the zeros of the polynomials

$$R_n(x,\lambda)=\frac{\partial}{\partial\lambda}P_n(x,\lambda);\quad S_{n-2}(x,\lambda)=\frac{\partial}{\partial\lambda}\{k_n^{-1}P_n(x,\lambda)\}.$$

The positive zeros of R_n and S_{n-2} alternate with the positive zeros of P_n. In the further part of the paper a necessary and sufficient condition is obtained for the reality of the zeros of $R_n(x,\lambda)+c(\lambda)P_n(x,\lambda)$ where $c(\lambda)$ is a given function of λ. Analogous considerations are indicated for the Jacobi polynomials (depending on two parameters). *G. Szegő.*

Carlitz, L. The multiplication formulas for the Bernoulli and Euler polynomials. Math. Mag. 27, 59–64 (1953).

The main object of the present note is to prove the following theorem. Let k be a fixed integer >1 and let $\alpha_{k1},\cdots,\alpha_{kk}$ denote complex numbers such that $\alpha_{k1}+\cdots+\alpha_{kk}=1$; let $|\lambda_k|\neq 1$ or 0 and let $\beta_{k1},\cdots,\beta_{kk}$ be distinct numbers. Then the equation

$$\sum_{r=1}^{k}\alpha_{kr}f_m(t+\beta_{kr})=\lambda_k^{-m}f_m(\lambda_k t)$$

is satisfied by a unique set of normalized polynomials $\{f_m(t)\}$ which form an Appell set. *A. L. Whiteman.*

Bhatia, A. B., and Wolf, E. On the circle polynomials of Zernike and related orthogonal sets. Proc. Cambridge Philos. Soc. 50, 40–48 (1954).

In the theory of diffraction F. Zernike has introduced certain (not necessarily real) polynomials $V(x,y)$ of the real variables x, y which are orthogonal on the unit circle $x^2+y^2\leqq 1$. The authors seek a systematic approach to these polynomials by the orthogonality relation

$$\iint_{x^2+y^2\leqq 1} V_\alpha(x,y)\overline{V_\beta(x,y)}dxdy=A_{\alpha\alpha}\delta_{\alpha\beta}$$

where $A_{\alpha\alpha}>0$, to which the condition is added that V should be invariant under a rotation about the origin. From this $V(r\cos\varphi, r\sin\varphi)=R(r)e^{il\varphi}$ follows where l is an integer and $R(r)$ a polynomial of degree n in which no power occurs lower than $|l|$. Moreover, $n-l$ is even. Assuming that the

system contains polynomials for all such values of n and l, only one eligible set remains. It can be easily characterized by orthogonalizing the functions $r^{|l|}$, $r^{|l|+2}$, $r^{|l|+4}$, $\cdots$ in $0 \leqq r \leqq 1$, and considering all $l=0, \pm 1, \pm 2, \cdots$. The resulting polynomials are of course essentially Jacobi (ultraspherical) polynomials. *G. Szegö* (Stanford, Calif.).

Parodi, Maurice. Sur une propriété des polynomes récurrents. C. R. Acad. Sci. Paris **237**, 1304–1305 (1953).

Let $P_n(x)$ be polynomials defined by the recurrence relation $f_1(n+1)P_n(x)+[\phi(n)-x]P_{n-1}(x)+f_2(n)P_{n-2}(x)=0$, $n \geqq 2$, $P_0=1$, $P_{-1}=0$, $\phi(k) \geqq 0$ for every integer k. If $f_1(k)$ and $f_2(k)$ have the same signs for every integer k, the polynomials $P_n(x)$ will have the same zeros as the polynomials $Q_n(x)$ defined by the recurrence relation

$$|f_1(n+1)|Q_n(x)+[\phi(n)-x]Q_{n-1}(x)+|f_2(n)|Q_{n-2}(x)=0, \quad n \geqq 2, \quad Q_0=1, \quad Q_{-1}=0.$$

In particular, if $\phi(k)=0$ for every k, this result reduces to the known property of the $P_n(x)$ that all their non-zero zeros are symmetric with respect to the origin.

E. Frank (Chicago, Ill.).

Butzer, P. L. Linear combinations of Bernstein polynomials. Canadian J. Math. **5**, 559–567 (1953).

Let $B_n(x)$ be the Bernstein polynomials associated with the function $f(x)$, $0 \leqq x \leqq 1$. The author is interested in linear combinations of $B_n(x)$ with different n which approximate $f(x)$ more closely than the $B_n(x)$ themselves. These combinations $L(n, 2k, x)$ are defined as follows:

$$L(n, 0, x)=B_n(x);$$
$$(2^k-1)L(n, 2k, x)=2^kL(2n, 2k-2, x)-L(n, 2k-2, x).$$

Assuming that $f(x)$ is bounded and $f^{(2k)}(x)$ exists at the point x, we have $L(n, 2k-2, x)-f(x)=O(n^{-k})$ and $L(n, 2k, x)-f(x)=o(n^{-k})$. Further approximations are discussed under the condition that $f^{(2k)}(x)$ exists and is continuous in the whole interval $[0, 1]$, having there a given modulus of continuity. Some of these results can not be improved. *G. Szegö* (Stanford, Calif.).

Clement, Preston R. The Chebyshev approximation method. Quart. Appl. Math. **11**, 167–183 (1953).

This paper gives a clear and detailed account of the Chebyshev approximation method and of the Chebyshev polynomials. Applications of Chebyshev polynomials to problems of engineering and physics are reviewed.

R. J. Duffin (Pittsburgh, Pa.).

Green, H. S., and Messel, H. On the expansion of functions in terms of their moments. Quart. Appl. Math. **11**, 403–409 (1954).

Two processes are discussed in order to represent a function $f(x)$ in terms of its moments $f_n=\int_{-\infty}^{\infty} f(x)x^n dx$. The first is as follows:

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}f_n\delta^{(n)}(x),$$

where the derivatives of the δ-function must be interpreted properly. The second representation employs a distribution function $w(x)$ and the associated orthonormal polynomials $\{S_n(x)\}$. Then $f(x)=w(x)\sum_{n=0}^{\infty}c_nS_n(x)$, where for c_n closed expressions can be found involving only the moments of $f(x)$ and $w(x)$. The last expansion has an analog for discrete distributions. Application is given to a partial differential equation with a certain boundary condition. *G. Szegö.*

Harmonic Functions, Potential Theory

Dinghas, Alexandre. Sur quelques théorèmes concernant la convexité des moyennes d'une classe des fonctions sousharmoniques. C. R. Acad. Sci. Paris **237**, 594–595 (1953).

The author considers functions $u(P)$, $P=P(x_1, \cdots, x_n)$, defined in the half-space $x_1>0$ and satisfying the conditions that $\lim\sup_{P \to Q} u(P) \leqq 0$, where $x_1=0$ at the point Q; that $u(P)$ is twice differentiable at each point of the set $E(u \geqq 0)$; that the set $E_1(u>0)$ is nonnull; and that at each point of E_1 we have $\Delta u \geqq cu$, where c is a nonnegative continuous function. Let S_r be the hemisphere with center at the origin, radius r, and $x_1>0$; set $E_r=E \cap S_r$, $x_1=r\cos\phi$,

$$\psi(P)=\frac{\overset{+}{u}(P)}{\cos\phi}, \quad m_k(r)=\left\{\int_{E_r}\psi^{2k}(P)\cos^2\phi\, d\omega_n\right\}^{1/k},$$

where $k \geqq 1$; and let $M_1(r)=\max\psi(P)$ on S_r. It is shown that for each function $u(P)$ of the foregoing class, $m_k(r)r^{n-2}$ is a convex function of r^n; m_k^k/r is a positive and nondecreasing function of r, and is a convex function of $1/r^n$; $M_1^2(r)r^{n-2}$ is a convex function of r^n; and $M_1(r)/r$ is a positive and nondecreasing function of r. Application is made to the classical Phragmen-Lindelöf theorem.

E. F. Beckenbach (Los Angeles, Calif.).

Helson, Henry. Note on harmonic functions. Proc. Amer. Math. Soc. **4**, 686–691 (1953).

Soit $u(r, \theta)=\sum_{n=-\infty}^{+\infty}a_nr^{|n|}e^{in\theta}$ une fonction harmonique de la classe de Poisson-Stieltjes dans le cercle-unité, v la fonction conjuguée de u; si les a_n ne prennent qu'un nombre fini de valeurs distinctes, $u+iv$ est une fonction rationnelle. Ce résultat, dont l'énoncé rappelle un théorème bien connu de Szegö [S.-B. Preuss. Akad. Wiss. **1922**, 88–91], est établi par une méthode toute différente, l'auteur se ramenant à déterminer les mesures de Radon sur le groupe des réels modulo 2π qui sont idempotentes pour le produit de composition. *J. Deny* (Strasbourg).

Ozawa, Mitsuru. Remarks on Mr. Ullemar's second harmonic measure. Kōdai Math. Sem. Rep. **1953**, 93–96 (1953).

Comments on a recent paper of L. Ullemar [Ark. Mat. **2**, 87–97 (1952); these Rev. **14**, 470]. Existence proof for second harmonic measure, relation between second harmonic measure and other domain functions. *M. Heins.*

Tsuji, Masatsugu. On the capacity of general Cantor sets. J. Math. Soc. Japan **5**, 235–252 (1953).

Let $k \geqq 2$ be a fixed integer, $\Delta(i_1)$ $(i_1=1, 2, \cdots, k)$ disjoint intervals contained in a given interval Δ, and generally

$$\Delta(i_1, i_2, \cdots, i_n) \subset \Delta(i_1, i_2, \cdots, i_{n-1}), \quad i_n=1, 2, \cdots, k.$$

We define

$$E=\prod_{n=1}^{\infty}\sum_{1 \leqq i_n \leqq k}\Delta(i_1, i_2, \cdots, i_n).$$

Moreover, we assume the existence of two positive numbers a, b such that ($|\Delta|$ is the length of Δ)

$$|\Delta(i_1, \cdots, i_{n-1}, \nu)| \geqq a|\Delta(i_1, \cdots, i_{n-1})|, \quad \nu=1, 2, \cdots, k,$$

and the distance of $\Delta(i_1, \cdots, i_{n-1}, \mu)$ and $\Delta(i_1, \cdots, i_{n-1}, \nu)$ is $\geqq b|\Delta(i_1, \cdots, i_{n-1})|$ for all $\mu \neq \nu$. It is shown that the linear measure of E is zero and the logarithmic capacity γ of E satisfies the inequality $\gamma \geqq b|\Delta| \exp[(\log a)/(k-1)]$. Moreover, every point of E is a regular point for Dirichlet's

problem. The definitions and results are quite similar in plane and in space (circles and spheres are used in place of intervals), except that in space we have the inequality $\gamma \geqq bR(ak-1)/a(k-1)$, where R is the radius of the initial sphere and $ak>1$ is assumed. *G. Szegö.*

Walsh, J. L. Note on the shape of level curves of Green's function. Duke Math. J. **20**, 611–615 (1953).

Une région R est dite avoir la propriété T si $w \in R$ entraîne $T(w) \in R$. Etude des transformations T telles que la propriété T pour R entraîne la même propriété pour chaque région R_r définie par $G(w)>-\log r$, $G(w)$ étant la fonction de Green de R admettant le point intérieur $w=0$ pour pole. Extension au cas où, R étant simplement connexe, R_r est l'image d'un horicycle dans la représentation conforme de R sur $|z|<1$. Cette étude très simple généralise un résultat de L. R. Ford [même J. **1**, 103–104 (1935)] correspondant au cas $T(w)=Aw$ ($A=$Cte.). [Note de l'auteur: Theorem 2 treats for a single transformation a geometric situation previously treated for a set of transformations by J. L. Wolff [Nederl. Akad. Wetensch., Proc. **33**, 1023–1024 (1930)], although Wolff uses an additional (and unnecessary) hypothesis.] *J. Lelong-Ferrand* (Lille).

Lesky, P. Über eine besondere Klasse von zweiten Randwertaufgaben. Österreich. Ing.-Arch. **7**, 231–236 (1953).

The author considers for the elliptical region in the plane, the second boundary-value problem of logarithmic potential, in which the directional derivative in a given direction is assigned on the perimeter, and the given direction also is allowed to vary in a piecewise continuous, but otherwise arbitrary, manner along the perimeter. The solution is obtained in formal fashion by assuming a Fourier series in terms of confocal elliptical coordinates. The special case of a circle and a direction rotating twice as fast as the normal is considered in some detail, and recursion formulas for the Fourier coefficients are found in this case. The related convergence questions are studied by Platone [Ann. Scuola Norm. Super. Pisa (3) **5**, 57–70 (1951); these Rev. **13**, 601] and by Lesky [ibid. **6**, 255–267 (1953); these Rev. **14**, 1145]. *J. W. Green* (Los Angeles, Calif.).

Komatu, Yûsaku. Integralformel betreffend Neumannsche Randwertaufgabe für einen Kreisring. Kōdai Math. Sem. Rep. **1953**, 37–40 (1953).

Représentation explicite de la solution du problème de Neumann pour la couronne $C: \rho<|z|<1$. $V(z)$ harmonique dans C, satisfaisant à

$$\partial V(e^{i\phi})/\partial n_i = P(\phi), \quad \partial V(\rho e^{i\phi})/\partial n_i = Q(\phi),$$

avec la condition nécessaire $\int_0^{2\pi}[P(\phi)+\rho Q(\phi)]d\phi=0$, est calculé par une somme d'intégrales où les noyaux $K(z,\phi)$ s'expriment à partir de fonctions elliptiques de $w=i\log z+\phi$. *P. Lelong* (Lille).

Beckert, Herbert. Eine Eigenschaft der klassischen Greenschen Funktionen erster und zweiter Art. Math. Nachr. **10**, 55–61 (1953).

In this paper, a new mode of representing a function in terms of its partial derivatives and a corresponding new form of the associated compatibility relations are derived. Applications are then indicated to a number of problems as, for instance, to that of the uniform convergence of functions whose partial derivatives converge in an L^p sense ($p>0$) over a given region. The paper deals chiefly with the case of a function defined in a simply connected plane domain D which is of class $\mathfrak{B}$ in the sense of Lichtenstein. Use is made of the classical Green's function for D of the first and second kinds, which are harmonic functions $G^I(x,y;\xi,\eta)$, $G^{II}(x,y;\xi,\eta)$ having logarithmic singularities at (ξ,η) and satisfying the boundary conditions $G^I(P;\xi,\eta)=0$, $\partial G^{II}(P;\xi,\eta)/\partial n_P=0$ at the points P of the boundary of D, n_P being the normal to the boundary at P; G^{II} is assumed to be so normalized that $\iint_D G^{II}(x,y;\xi,\eta)d\xi d\eta=0$. The principal result of the paper is the following theorem: Let $p(x,y)$, $q(x,y)$ be bounded, measurable functions in the closure of D. Then there exists a Lipschitz-continuous function $w(x,y)$ for which $w_x=p$, $w_y=q$ almost everywhere in D, if, and only if, the condition

$$\iint_D \left\{-\frac{\partial G^I(x,y;\xi,\eta)}{\partial x}q(x,y) + \frac{\partial G^I(x,y;\xi,\eta)}{\partial y}p(x,y)\right\}dxdy=0$$

is satisfied. In case this condition is satisfied, $w(\xi,\eta)$ is determined up to an additive constant as

$$w(\xi,\eta)=\frac{1}{2\pi}\iint_D \left\{\frac{\partial G^{II}}{\partial y}(x,y;\xi,\eta)q(x,y) + \frac{\partial G^{II}}{\partial x}(x,y;\xi,\eta)p(x,y)+\frac{2\pi}{F}w(x,y)\right\}dxdy+\text{const.},$$

where F is the area of D. *A. Douglis.*

Hitotumatu, Sin. Note on the Dirichlet problem. Comment. Math. Univ. St. Paul. **2**, 13–21 (1953).

L'auteur complète d'abord la théorie du potentiel, en suivant l'exposé de H. Cartan, sur le petit point suivant: soit f finie continue sur la frontière d'une ensemble borné A de R^n (par exemple $n\geqq 3$) et μ^P les masses obtenues par balayage de $+1$ en P relativement à A (balayage de type extérieur ou intérieur). Alors $\int f d\mu^P$ tend vers $f(Q)$ en tout "régulier" Q de A d'après les propriétés de μ^P, et est harmonique à l'extérieur de A. On le voit en approchant f par un potentiel; puis on utilise l'harmonicité relative à P du potentiel de μ^P en un point fixe P_0; cela vient de l'égalité avec le potentiel de μ^{P_0} en P.

Puis l'auteur s'occupe, indépendamment, du problème de Dirichlet dans le cercle, d'une façon qui, malgré un lapsus, semble suivre une voie importante. Etant donnée une distribution de Schwartz T sur la variété qu'est la circonférence unité Γ, il existe une fonction harmonique U unique dans le cercle, dont la trace sur la circonférence concentrique de rayon r est une fonction de l'angle polaire θ, qui converge vers T au sens des distributions sur Γ. Cela résulte du cas particulier où T est la masse 1 en un point Q, U étant alors le noyau de Poisson $(2\pi)^{-1}(1-\overline{OM}^2)/\overline{MQ}^2$. Le problème de Neumann du cercle énoncé avec une distribution se ramène au problème précédent. *M. Brelot* (Grenoble).

Mogilevskiĭ, Š. I. On the stability of the Dirichlet problem. Doklady Akad. Nauk SSSR (N.S.) **78**, 1093–1096 (1951). (Russian)

Let D be a domain in three-space with compact boundary, E an F_σ-set in the complement of D. The Dirichlet problem is called stable in D with respect to E if the following statement is true. Let ϕ be a continuous function defined everywhere, u the (generalized) solution of the Dirichlet problem for the domain D and boundary values ϕ. Let $\{F_n\}$ be any sequence of closed sets, $F_n \subset F_{n+1} \to E$, and let D_n be that

component of the complement of F which contains D. Let u_n be the (generalized) solution of the Dirichlet problem for the domain D_n and boundary values ϕ. Then $u_n \to u$.

The author attributes this formulation, generalizing the one due to M. V. Keldyš and M. A. Lavrent'ev [Izvestiya Akad. Nauk SSSR. Ser. Mat. **1937**, 551–595] to P. P. Korovkin [Doklady Akad. Nauk SSSR (N.S.) **61**, 781–784 (1948); these Rev. **10**, 297]. He states several necessary and sufficient conditions for stability, generalizing results of Keldyš and Lavrent'ev and of M. Inoue [Proc. Imp. Acad. Japan **14**, 273–277 (1938)]. *L. Bers.*

Šilov, G. E. On doubly periodic vectorially smooth functions. Ukrain. Mat. Žurnal **4**, 25–35 (1952). (Russian)

Dans un mémoire antérieur [Uspehi Matem. Nauk (N.S.) **6**, no. 5(45), 176–184 (1951); ces Rev. **14**, 374], l'A. étudie les champs plans de vecteurs $\{u(x, y), v(x, y)\}$, assez réguliers dans un domaine pour y admettre une divergence et un rotationnel au sens généralisé. Donnons-nous alors deux fonctions réelles continues, $\mu(s, t)$, $\nu(s, t)$, périodiques (de période 2π) en s et en t, telles que:

$$\int_0^{2\pi}\int_0^{2\pi} \mu ds dt = \int_0^{2\pi}\int_0^{2\pi} \nu ds dt = 0.$$

L'auteur montre qu'il existe un champ $\{u, v\}$ et un seul, défini à une constante additive près, tel que: div $(u, v) = \mu$; rot $\{u, v\} = \nu$. *J. Kravtchenko* (Grenoble).

***Manacorda, Tristano. Sulle discontinuità delle derivate del potenziale di doppio strato.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 516–525. Casa Editrice Perrella, Roma, 1953.

Une méthode utilisée par P. Burgatti [Analisi vettoriale generale e applicazioni, v. 3, Zanichelli, Bologna, 1931] pour le calcul effectif des discontinuités des dérivées secondes d'un potentiel de double-couche, est reprise et appliquée au cas des dérivées troisièmes. On étudie ensuite les discontinuités en un point du bord de la surface qui porte la double-couche, et différentes sortes de singularités.

J. Deny (Strasbourg).

Moon, Parry, and Spencer, Domina Eberle. The meaning of the vector Laplacian. J. Franklin Inst. **256**, 551–558 (1953).

Differential Equations

Sawyer, Walter Warwick. On polynomial sequences connected with differential equations. Math. Nachr. **9**, 269–280 (1953).

Continuing an investigation due to Hahn [Math. Nachr. **4**, 1–11 (1951); these Rev. **13**, 233] the author shows how polynomial chains (see Hahn) can be constructed which satisfy a sequence of linear homogeneous second-order differential equations of the Fuchsian type. The question of the number of apparent singularities in these equations is also considered, as well as a method for determining the apparent singularities. Reference should be made to a later paper of Hahn [Math. Nachr. **7**, 85–104 (1952); these Rev. **14**, 556]. *E. A. Coddington* (Los Angeles, Calif.).

Halanaĭ, A. On a linear differential equation with an almost periodic coefficient. Doklady Akad. Nauk SSSR (N.S.) **88**, 419–422 (1953). (Russian)

Consider the equation $x'' + px = 0$, where

$$p(t) = \mu + \sum_{k=1}^{\infty} a_k \exp(i\alpha_k t), \quad \alpha_k > \alpha > 0,$$

and $\sum |a_k| < \infty$. It is proved that there exists a solution of the form $x(t) = \exp(i\mu^{\frac{1}{2}}t)\Phi(t)$, where $\Phi(t) = \sum_{k=0}^{\infty} b_k \exp(i\beta_k t)$, $\beta_k = \sum l_i \alpha_i$, the l_i being non-negative integers, and $\beta_0 = 0$. If $2\mu^{\frac{1}{2}}$ is not representable as $\sum l_i \alpha_i$, where the l_i are as before, then a second solution has the form $x(t) = \exp(-i\mu^{\frac{1}{2}}t)\psi(t)$, where $\psi(t) = \sum_{k=0}^{\infty} c_k \exp(i\beta_k t)$; whereas if $2\mu^{\frac{1}{2}} = \sum l_i \alpha_i$, then a second solution is of the form

$$x(t) = t \exp(i\mu^{\frac{1}{2}}t)\Phi_1(t) + \exp(i\mu^{\frac{1}{2}}t)\Phi_2(t) + \exp(-i\mu^{\frac{1}{2}}t)\Phi_3(t),$$

where $\Phi_j(t) = \sum_{k=0}^{\infty} d_k^j \exp(i\beta_k t)$, $j = 1, 2, 3$. This generalizes a result obtained by Putnam and Wintner [Amer. J. Math. **73**, 792–806 (1951); these Rev. **13**, 557] for the case $\mu = 0$. The proof follows by successive approximations.

E. A. Coddington (Los Angeles, Calif.).

Kimura, Toshifusa. Sur une généralisation d'un théorème de Malmquist. Comment. Math. Univ. St. Paul. **2**, 23–28 (1953).

The author generalizes a theorem of Malmquist [Acta Math. **36**, 297–343 (1913)] relative to the equation (1) $dy/dx = P(x, y)/Q(x, y)$, where P, Q are polynomials in x, y. The following is established. (I) Consider the equation (2) $x^{\sigma+1}dy/dx = P/Q$, where σ is an integer ($\geqq 0$) and P, Q are polynomials in y whose coefficients are analytic in x at $x = 0$; if S is the set of accumulation of a solution $y(x)$ (near $x = 0$), then S consists of just a single point or S is the whole plane. (II) Every solution of (1) takes on all the values, except for a finite number of values, in an arbitrary neighborhood of the essentially singular point. (III) Under conditions as in (I), let $P = P_0(x)\prod_1^n (y - P_i(x))$, $Q = Q_0(x)\prod_1^n (y - Q_i(x))$, $\alpha_i = P_i(0)$, $\beta_i = Q_i(0)$, and assume that there exists a β_i distinct from all the α_i; if a solution $y(x)$ of (2) has no critical point in a vicinity of the origin, except at the origin, then the latter is an ordinary singular point. *W. J. Trjitzinsky* (Urbana, Ill.).

Reĭzin' [Reizinš], L. È. Behavior near to a singular point of integral curves of a system of three differential equations. Latvijas PSR Zinātņu Akad. Vēstis 1951, no. 2 (43) 333–346 (1951). (Russian. Latvian summary)

An analysis is made of the solutions of differential equations $\dot{x}_i = F_i(x_1, x_2, x_3)$ $(i = 1, 2, 3)$ near the singular point O: $(0, 0, 0)$. It is assumed that the F_i differ from homogeneous polynomials of a fixed degree m by terms which are $o(r^m)$, where r is distance from O. The equations are rewritten in spherical coordinates r, ϕ, θ in the form:

$$\dot{r} = r^m R(\phi, \theta) + h(r, \phi, \theta), \quad r\dot{\phi} = r^m \Phi(\phi, \theta) + f(r, \phi, \theta),$$
$$r \sin\phi\,\dot{\theta} = r^m \Theta(\phi, \theta) + g(r, \phi, \theta).$$

An exceptional direction at O is defined to be one for which the corresponding (ϕ, θ) are such that Φ and Θ are 0. The following theorems are proved. 1. At least one exceptional direction exists. 2. If $\Phi = \Theta = 0$ and a continuous function $B(r)$ exists, whose integral from O to some positive r_0 exists, such that $|f| \leqq B(r)$ and $|g \csc\phi| \leqq B(r)$ for $r < r_0$, then all integral curves near O end at O with a definite direction and there is an integral curve for every direction at O. If further the functions hB^{-1}, fB^{-1} and $gB^{-1}\csc\phi$ satisfy Lipschitz conditions with respect to ϕ and θ, then there is precisely

one curve for each direction. 3. If the direction of the z-axis is an isolated exceptional direction and $K=\Phi/R>0$ in a sufficiently narrow conical region $S: 0\leqq\phi\leqq\delta$, $0<r\leqq r_0$ about the z-axis, then all solutions entering S for $r>0$ continue in S and end at 0 with the z-axis as tangent. 4. If $K<0$ in the set S, then there exists at least one solution entering S for $r=r_0$ and ending at 0 with the z-axis as tangent but there also exist other solutions entering S for $r=r_0$ and then leaving S for $r<r_0$. *W. Kaplan* (Ann Arbor, Mich.).

Kestin, J., and Zaremba, S. K. Geometrical methods in the analysis of ordinary differential equations. Introduction to non-linear mechanics. Appl. Sci. Research B. **3**, 149–189 (1953).

This is an exposition of the qualitative theory of the real, plane, differential system $\dot{x}=f(x, y)$, $\dot{y}=g(x, y)$, including such topics as: (a) behavior of solutions near a singular (critical) point, both elementary and of higher order; (b) Kronecker-Poincaré index of an isolated critical point; (c) Poincaré and Bendixson criteria for the existence of periodic solutions; (d) extension of the plane differential system to the compact sphere or the projective plane. Several examples referring to one-dimensional adiabatic gas flow are analysed by these classical techniques.

L. Markus (New Haven, Conn.).

Ascoli, Guido. Sul comportamento asintotico degli integrali dell'equazione $y''=(1+f(t))y$ in un caso notevole. Rivista Mat. Univ. Parma **4**, 11–29 (1953).

The paper contains a systematic asymptotic theory of the differential equation $y''=(1+f(t))y$, as $t\to+\infty$, under the sole assumption that $|f|^p$ is integrable, in $0\leqq t<\infty$, for some $p\geqq 1$. The central result is the following: Let the functions w_n be defined by

$$w_1'+2w_1=f;\ w_n'+2w_n=-\sum_1^{n-1}w_sw_{n-s},\ n>1;\ \text{and}\ w_n(0)=0.$$

Then the differential equation in the title possesses a solution for which

$$\log y=t+\sum_1^{\nu-1}\int_0^t w_s(\tau)d\tau+C+o(1),$$

where $\nu-1<p\leqq\nu$. A subsequent asymptotic calculation of the $\int_0^t w_s(\tau)d\tau$ leads to a somewhat different "canonical" form, which, although more involved in appearance, is more suitable for the calculation of explicit asymptotic formulas for given moderate values of p. For $1<p\leqq 2$ and for $2<p\leqq 3$ the formulas so obtained coincide with those proved, under more restrictive conditions, by P. Hartman [Trans. Amer. Math. Soc. **63**, 560–580 (1948); these Rev. **9**, 589] and by R. Bellmann [Ann. Mat. Pura Appl. (4) **31**, 83–91 (1950); these Rev. **13**, 132] respectively.

The tools used by the author are essentially similar to those employed in the papers cited above. But the arguments are greatly simplified by a systematic use of the general properties of the operator $V_\alpha(t)=\int_0^t e^{-\alpha(t-\tau)}f(\tau)d\tau$.

W. Wasow (Los Angeles, California).

Ascari, Aldo. Studio asintotico di un'equazione relativa alla dinamica del punto. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **16(85)**, 278–288 (1952).

The existence of periodic solutions of the differential equation $\ddot{x}+x+\varphi(\dot{x})=f(t)$ with periodic $f(t)$ has been studied by several authors [e.g., Caccioppoli and Ghizzetti, Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 493–501 (1942); these Rev. **6**, 153; Levinson, J. Math. Physics **22**, 181–187 (1943); these Rev. **5**, 183]. The conditions imposed by these authors imply that $\varphi(\dot{x})$ must be unbounded, as $\dot{x}\to\pm\infty$. In the present paper the existence of at least one solution with the same period as $f(t)$ is proved assuming (besides the usual smoothness requirements) only that

$$\liminf_{y\to+\infty}\varphi(y)>\max|f(t)|,\quad \limsup_{y\to-\infty}\varphi(y)<-\max|f(t)|.$$

If, in addition, $\varphi(y)$ is increasing for $|y|\leqq R_0$, where R_0 depends on the properties of $\varphi(y)$ and $f(t)$, then the periodic solution is unique and stable. The method of the proof is topological. *W. Wasow* (Los Angeles, Calif.).

Gagliardo, Emilio. Sul comportamente degli integrali dell'equazione differenziale non lineare

$$x''+f(x)x'+g(x)=0$$

con $g(x)$ crescente e $f(x)$ positiva per $|x|>M>0$. Boll. Un. Mat. Ital. (3) **8**, 309–314 (1953).

To the assumptions stated in the title add that $f(x)$, $g(x)$ must be continuous for all x, and that $g(0)=0$, $g'(0)>0$. Then it is shown that for sufficiently large t every solution of the differential equation either tends monotonically to zero, as $t\to\infty$, or oscillates (i.e., has infinitely many changes of sign). If $f(0)<2\sqrt{g'(0)}$ all solutions oscillate; if $f(x)\geqq 0$ and $f(0)>2\sqrt{g'(0)}$ they all show the former behavior. If $f(x)\geqq 0$ and $f(0)>0$ all solutions tend to zero (possibly oscillating). The proofs are direct and elementary.

W. Wasow (Los Angeles, Calif.).

Zlámal, Miloš. Asymptotische Eigenschaften der Lösungen linearer Differentialgleichungen. Math. Nachr. **10**, 169–174 (1953).

The coefficients of the differential equation

$$y^{(n)}+\sum_{j=1}^{n}a_j(x)y^{(n-j)}=f(x)$$

are assumed to be continuous for all $x\geqq x_0$. It is shown that, under suitable conditions, the solutions of this differential equation are asymptotically, as $x\to+\infty$, equal to those of $y^{(n)}=0$. One result is as follows: If for $\epsilon\geqq 0$, $\int^\infty t^{j+n-2+\epsilon}|a_j(t)|dt<\infty$ $(j=1,\cdots,n)$, $\int^\infty t^{n-1+\epsilon}|f(t)|dt<\infty$, then the general solution is of the form

$$y(x)=\sum_{j=0}^{n-1}c_jx^j+o(x^{-\epsilon}).$$

If only the weaker inequalities $\int^\infty t^{j-1+\epsilon}|a_j(t)|dt<\infty$ are required of the $a_j(t)$, the asymptotic relationship must be weakened to $y(x)=\sum_{j=0}^{n-1}c_jx^j[1+o(x^{-\delta})]+o(x^{-\delta})$, $\delta<\epsilon$. The proofs employ a variant of the well-known technique for establishing the asymptotic equality of the solutions of two asymptotically similar differential equations by means of a Volterra integral equation for the difference of those solutions. *W. Wasow* (Los Angeles, Calif.).

Schäfke, Friedrich W. Einige Stabilitätskriterien. Z. Angew. Math. Mech. **33**, 283–285 (1953).

Let $f(x)$ be a continuous non-constant periodic function with the period π. It is shown that all of the solutions of the equation $y''(x)+f(x)y(x)=0$ are stable provided $f(x)$ satisfies one of the following conditions. (1) There exist a constant α and a non-negative integer n such that $f(x)\geqq\alpha^2$, $n^2\leqq\alpha^2<(n+1)^2$, and

$$\frac{1}{\pi}\int_0^\pi f(x)dx\leqq\alpha^2+\frac{2(n+1)\alpha}{\pi}\cot\frac{\alpha\pi}{2n+2}.$$

(2) There exist a constant α and a non-negative integer n such that $f(x) \leqq \alpha^2$, $n^2 < \alpha^2 \leqq (n+1)^2$, and $\pi^{-1}\int_0^\pi f(x)dx \geqq n\alpha$.
L. A. MacColl (New York, N. Y.).

Tumura, Masamitu. The exact criterion for the stability of non-linear vibrations. Tech. Rep. Osaka Univ. 1, nos. 1–9, 35–50 (1951).

The author considers the equation (1) $\ddot{x}+f(x,\dot{x})=c(t)$ where f is analytic in a region R of the $(x,\dot{x})$-phase plane and $c(t)=\sum_{n=-\infty}^{+\infty} c_n e^{int}$, $c_n=\bar{c}_{-n}$, is a real, periodic analytic function. Assuming the existence of a real periodic, analytic solution $x=\xi(t)=\sum_{n=-\infty}^{\infty} a_n e^{int}$, $a_n=\bar{a}_{-n}$, lying in R, the author studies the stability (in the sense of Liapounoff) of $x=\xi(t)$ by means of the characteristic roots of the variational equation based on $x=\xi(t)$. Theorem. A necessary and sufficient condition that $x=\xi(t)$ is stable (both characteristic roots negative) is that

$$\int_0^{2\pi/\omega} \frac{\partial f}{\partial \dot{x}}(\xi(t),\dot{\xi}(t))dt>0, \quad H(0)>0, \quad H(\tfrac{1}{2}i\omega)<0,$$

where $H(\mu)$ is Hill's determinant for the variational equation. The author finds a number of properties of $H(\mu)$ in special cases including that of the damped Mathieu equation.
L. Markus (New Haven, Conn.).

Tumura, Masamitu. The exact criterion for the stability of nonlinear vibrations. II. Tech. Rep. Osaka Univ. 2, nos. 25–38, 1–10 (1952).

Assuming the existence of almost periodic solutions

$$x=\xi(t,\varphi)=\sum_{m,n=-\infty}^{\infty} a_{m,n}\exp\left[ima(t+\varphi)+in\omega t\right], \quad a_{m,n}=\bar{a}_{-m,-n}$$

and $-\infty<\varphi<\infty$ for the equation $\ddot{x}+f(x,\dot{x})=c(t)$ where f is analytic and $c(t)$ periodic [see above review], the author studies the stability of $x=\xi(t,\varphi)$ by means of the characteristic exponent (the one which corresponds to the solution $\partial\xi/\partial\varphi$ is zero) of the variational equation. Theorem. A necessary and sufficient condition that, for each φ, $x=\xi(t,\varphi)$ be stable (one characteristic root negative and one zero) is

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{\partial f}{\partial \dot{x}}(\xi(t),\dot{\xi}(t))dt>0.$$

Since the variational equation has an almost periodic matrix, some further discussion should be added to relate the characteristic exponents to the Liapounoff stability of $x=\xi(t,\varphi)$.
L. Markus (New Haven, Conn.).

Aymerich, Giuseppe. Sulle oscillazioni autosostenute impulsivamente. Rend. Sem. Fac. Sci. Univ. Cagliari 22 (1952), 34–37 (1953).

The differential system $\dot{x}=-\epsilon f(x)+y$, $\dot{y}=-\omega^2 x$, with $f(x)=2\omega[x-x_0\,\mathrm{sgn}\,(x)]$ describes the straight-line motion $x(t)$ of a point under the influence of a linear restoring force and linear damping modified by a numerically constant jump of the velocity whenever $x=0$. The author studies this motion for small ϵ by means of a slight modification of the approximation method of Krylov and Bogoliubov [Introduction to non-linear mechanics, Princeton, 1943; these Rev. 4, 142], and finds results essentially equivalent to those of Y. Rocard [Dynamique générale des vibrations, Masson, Paris, 1943; these Rev. 10, 628].
W. Wasow.

***Vogel, Théodore. Topologie des oscillations à déferlement.** Actes du Colloque International des Vibrations non linéaires. Ile de Porquerolles, 1951, pp. 237–256, Publ. Sci. Tech. Ministère de l'Air, Paris, no. 281 (1953).

A pair of first order, real, autonomous, ordinary differential equations defined in the plane with a "law of exchange" applied whenever a trajectory intersects a prescribed "discontinuity curve" is called a "déferlant" system. The author analyses déferlant systems for the existence and stability of periodic motions when the discontinuity curve (usually a simple closed curve) and the pair of differential equations are of certain special simple types. The discussion is phrased geometrically without precise statements on the general situation. The paper is motivated by an example of an electric multivibrator and photographs of oscillographs are compared with the corresponding mathematical analysis.
L. Markus (New Haven, Conn.).

Vogel, Théodore. Sur les systèmes déferlants. Bull. Soc. Math. France 81, 63–75 (1953).

The author reviews and extends his work on déferlant systems [see preceding review] in a more compressed notation. The extension consists in a more careful analysis of the cases in which the differential equations of the system have elementary critical points.
L. Markus.

***Bellin, Albert I. Non-autonomous systems.** Advances in Applied Mechanics, vol. 3, pp. 295–320. Academic Press Inc., New York, N. Y., 1953. $9.00.

Expository survey of recent work on nonlinear forced oscillations, concerned with topological and analytical methods for discussing the existence and stability of periodic solutions of $\ddot{x}=F(x,\dot{x},t)$.
G. E. H. Reuter.

***Stoker, J. J. Oscillations périodiques des systèmes non linéaires ayant une infinité de degrés de liberté. Periodic oscillations of nonlinear systems with infinitely many degrees of freedom.** Actes du Colloque International des Vibrations non linéaires, Ile de Porquerolles, 1951, pp. 61–74; discussion, p. 75, Publ. Sci. Tech. Ministère de l'Air, Paris, no. 281 (1953). (French and English)

The author proposes to extend the Poincaré small-parameter theory of periodic solutions to partial differential equations such as $u_{xx}-u_{tt}=\epsilon f(u,x,t)$. He points out that there may be difficulties (e.g., "small divisors") in proving the congruence of perturbation schemes. As an example, a first approximation to a solution of period $2\pi/\omega$ is calculated (heuristically) for the Duffing type equation

$$u_{xx}-u_{tt}+\epsilon(u+\beta u^3)=\epsilon F\sin x\cos\omega t,$$

when $\omega-1=O(\epsilon)$.
G. E. H. Reuter (Manchester).

***Cohen, Hirsh G. Synchronisation sous-harmonique dans le cas d'oscillations forcées conformes à l'équation de Van der Pol. Subharmonic synchronization for the forced Van der Pol equation.** Actes du Colloque International des Vibrations non linéaires, Ile de Porquerolles, 1951, pp. 169–187; discussion, p. 186, Publ. Sci. Tech. Ministère de l'Air Paris, no. 281 (1953). (French and English)

Solutions of period $2\pi/\omega$ of

$$\ddot{u}+\epsilon(u^2-1)\dot{u}+\omega_0^2 u=A\cos(3\omega t+\phi),$$

where ϵ is small and $\omega^2-\omega_0^2=c\epsilon$, are calculated by a perturbation method. A response diagram exhibits the amplitudes and stability character of these third-order subharmonics.
G. E. H. Reuter (Manchester).

Minorsky, Nicolas. Sur l'excitation asynchrone. C. R. Acad. Sci. Paris **237**, 964–966 (1953).

This note gives a brief sketch of a theory of an effect, occurring in some non-linear systems, in which a periodic oscillation is excited by a sinusoidal applied force having an arbitrary, but not too large, frequency. A more complete exposition will be required in order to make the theory entirely clear. *L. A. MacColl* (New York, N. Y.).

Hayes, Wallace D. On the equation for a damped pendulum under constant torque. Z. Angew. Math. Physik **4**, 398–401 (1953).

There exists a function $\alpha_0(\theta_0)$ such that $\alpha<\alpha_0(\theta_0)$ is a necessary and sufficient condition for the equation

$$\frac{d^2\theta}{dt^2}+\alpha\frac{d\theta}{dt}+\sin\theta-\sin\theta_0=0$$

to possess a solution for which $d\theta/dt$ is always positive and periodic with respect to θ. In this note the author gives the following bounds for $\alpha_0(\theta_0)$: $\alpha_1<\alpha_0(\theta_0)<2\sin(\theta_0/2)$, where $\alpha_1^2=(3\cos^2\theta_0+1)^{1/2}-2\cos\theta_0$. *L. A. MacColl.*

Conte, S. D., and Sangren, W. C. An asymptotic solution for a pair of first order equations. Proc. Amer. Math. Soc. **4**, 696–702 (1953).

As a prerequisite for obtaining expansion theorems of the Sturm-Liouville type for the set of equations

$$u'=(\lambda a(x)+b(x))v,\quad v'=(\lambda c(x)+d(x))u$$

in the interval $x\geqq 0$, the asymptotic behaviour of a solution u, v with prescribed values at the origin is investigated. It is assumed that a and c are positive. An expansion theorem for a finite interval is stated. The equations have the Dirac relativistic wave equations as a model. *L. Gårding.*

Bergmann, Howard G. The boundary layer problem for certain non-linear ordinary differential equations. Compositio Math. **11**, 119–169 (1953).

The author studies the boundary-layer effect in connection with the boundary-value problem

$$(1)\qquad p_{xx}=q^2/2,\quad kq_{xx}+pq=0,\quad k>0,\quad -1\leqq x\leqq 1,$$
$$p(-1)=p_1,\quad p(1)=p_2,\quad q_x(-1)=q_x(1)=0$$

as the parameter $k\to 0$. In case $p_1>0$ the transformation $t=(x+1)(p_1/k)^{1/2}$, $P=p/p_1$, $Q=k^{1/2}q/p_1$ converts (1) to a boundary-value problem on the interval $0\leqq t\leqq a=2(p_1/k)^{1/2}$. As $k\to 0$ there arises the asymptotic problem (2) $P_{tt}=Q^2/2$, $Q_{tt}+PQ=0$, $0\leqq t<\infty$, $P(0)=1$, $Q_t(0)=0$ with $Q(t)$ required to be continuous and its derivatives required to satisfy suitable integrability conditions on the interval $0\leqq t<\infty$. This asymptotic problem is the same as that treated by Friedrichs and Stoker [Amer. J. Math. **63**, 839–888 (1941); these Rev. **3**, 223] and results obtained by them are used in part in the present paper. The methods of the present paper parallel those of Friedrichs and Stoker, but in addition the author supplies a proof of the convergence of certain power series for $P(t)$ and $Q(t)$ which were introduced by Friedrichs and Stoker in connection with the asymptotic problem (2). If $q^k(x)$ and $p^k(x)$ denote the solution of (1), then the main results stated by the author may be summarized as follows: If both $p_1<0$ and $p_2<0$, then

$$q^k(x)=0,\quad p^k(x)=\tfrac{1}{2}[(p_2-p_1)x+(p_2+p_1)]$$

for all $k>0$; if either $p_1>0$ or $p_2>0$ or both, then $k^{1/2}q^k(x)\to 0$, $p^k(x)\to p^0(x)$ uniformly in every interval $-1<x_-\leqq x\leqq x_+<1$ as $k\to 0$, where $p^0(x)=\frac{1}{2}[(\gamma_2p_2-\gamma_1p_1)x+(\gamma_2p_2+\gamma_1p_1)]$ and $\gamma_i=1$ if $p_i<0$ and $\gamma_i=-.47271$ if $p_i>0$. That is, there is a boundary-layer effect if the boundary value is positive and otherwise not. The inequality (3.19) of the present paper is clearly misprinted, but even after this is corrected to what was evidently intended the inequality does not follow in the manner indicated by the author. Indeed, the first inequality of (3.20), which is deduced from (3.19), is not true in general for $k<1/2$ as the counterexample $q(x)=\sin n\pi x$ shows if n is sufficiently large. The reviewer did not determine what effect, if any, this error has on the validity of the results stated above. *C. E. Langenhop.*

Putnam, C. R. A sufficient condition for an infinite discrete spectrum. Quart. Appl. Math. **11**, 484–487 (1954).

The author applies an oscillation criterion proved earlier by himself [Duke Math. J. **16**, 633–636 (1949); these Rev. **11**, 437] to show that if $f(t)$ is real, continuous for $0\leqq t<\infty$ and $f(t)\to 0$ as $t\to\infty$ and if

$$\limsup_{h\to+0} h\left(\limsup_{S\to\infty} S\left[\limsup_{T\to\infty}\int_S^T f(t)\,dt + \int_0^S f(t)\,|(t-\gamma)/(S-\gamma)|^{\frac{1}{2}+\frac{1}{2}}dt\right]\right)>\tfrac{1}{4}$$

holds for every fixed $\gamma\geqq 0$, then the equation $x''+f(t)x=0$ is oscillatory. This in turn implies that the boundary-value problem $x''+(\lambda+f(t))x=0$, $x(0)\cos\alpha+x'(0)\sin\alpha=0$, $0\leqq\alpha<\pi$, x of class $L^2[0,\infty)$ has only a discrete spectrum for $\lambda<0$ with an infinity of eigenvalues clustering only at $\lambda=0$. An application is made to the two-particle problem in quantum mechanics to obtain a condition on the interaction potential sufficient to insure an infinite discrete spectrum of negative energy levels for a given value of the azimuthal quantum number 1. *C. E. Langenhop* (Ames, Iowa).

Moser, Jürgen. Störungstheorie des kontinuierlichen Spektrums für gewöhnliche Differentialgleichungen zweiter Ordnung. Math. Ann. **125**, 366–393 (1953).

Let L_ϵ denote the formal differential operator defined by $L_\epsilon u=-(pu')'+q_\epsilon u=\lambda ku$, where p, k are positive continuous functions on an open real interval $a<x<b$ (which may be unbounded), and $q_\epsilon(x)=\sum_{\nu=0}^{\infty}\epsilon^\nu q^{(\nu)}(x)$ for $|\epsilon|<\epsilon_0$ (ϵ real), where the convergence is uniform on every closed subinterval of $a<x<b$ and the $q^{(\nu)}$ are continuous except for certain discontinuity points (common for all ν). It is assumed that L_0 is in the limit circle case at a and limit point case at b. The author shows that under certain conditions on the perturbations $q^{(\nu)}$, $\nu=1,2,\cdots$, the operator L_ϵ will be in the limit circle case at a and limit point case at b for all sufficiently small $|\epsilon|$, and a boundary condition at a independent of ϵ may be prescribed which will allow a self-adjoint boundary value problem for L_ϵ to be defined in the space $L^2(a,b)$. Under an additional assumption on the unperturbed operator L_0 it is shown that the function of ϵ defined by $\rho_\epsilon(\Delta)=\rho_\epsilon(\lambda)-\rho_\epsilon(\mu)$, where $\Delta=[\mu,\lambda]$ is a closed real interval and ρ_ϵ is the spectral function, is regular, in a certain sense, for $|\epsilon|$ sufficiently small. A simple example indicates that some condition on L_0 is necessary for this regularity of ρ_ϵ. If A_ϵ denotes the self-adjoint operator in $L^2(a,b)$ naturally associated with L_ϵ and the boundary condition at a, and $E_\epsilon(\lambda)$ the corresponding family of projections appearing in the spectral decomposition of A_ϵ, then sufficient conditions for the regularity of $E_\epsilon(\Delta)$, as a function of ϵ, are given. It is further shown that under certain conditions A_ϵ is actually unitarily equivalent to the unperturbed

operator A_0 for small $|\epsilon|$. Applications of these results are given to various operators L_0, in particular the Hill equation on $0 \leq x < \infty$ and the operator $L_0 u = -u''$ on $0 \leq x < \infty$.

E. A. Coddington (Los Angeles, Calif.).

***Borg, Göran. Uniqueness theorems in the spectral theory of $y'' + (\lambda - q(x))y = 0$.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 276–287. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

Let L denote the differential operator defined by $Ly = -y'' + q(x)y$, where q is a continuous function on $0 \leq x < \infty$. It is assumed L is of the limit-point type at ∞. The boundary-value problem

$$Ly = \lambda y, \quad y(0) \cos \alpha + y'(0) \sin \alpha = 0$$

is considered in the space $L^2(0, \infty)$. For a given fundamental set of solutions of $Ly = \lambda y$ (λ a complex parameter) let $m(\lambda)$ be the limit point. It is shown that m (as a function of λ) determines q uniquely. If the spectra of the above problem and one obtained by replacing α by α^*, where $\alpha^* \neq \alpha \pmod{\pi}$, are discrete without finite accumulation points, then the eigenvalues of these spectra determine q uniquely. The spectral function ρ of the original problem also determines q uniquely. The results extend to the case where L is singular at zero also.

E. A. Coddington (Los Angeles, Calif.).

Marčenko, V. A. Some questions of the theory of one-dimensional linear differential operators of the second order. I, II. Trudy Moskov. Mat. Obšč. **1**, 327–420 (1952); **2**, 3–83 (1953). (Russian)

These papers are for the most part a detailed exposition of results previously announced [Doklady Akad. Nauk SSSR (N.S.) **72**, 457–460; **74**, 185–188, 657–660, 893–895 (1950); these Rev. **12**, 183, 502, 707, 698]. A systematic development, along with many applications, is given of the transformation operators which have the property of transforming solutions of one linear differential equation of the second order into those of another. Let $Lu = u'' - qu$ where q is a real function on $0 \leq x < a$ (or $-a < x < a$), $a = \infty$ not being ruled out, which is summable on each subinterval. Let $\omega_h = \omega_h(\lambda, x)$ be the solution of $Lu + \lambda u = 0$ satisfying $u(\lambda, 0) = 1$, $u'(\lambda, 0) = h$ if $h \neq \infty$, and $u(\lambda, 0) = 0$, $u'(\lambda, 0) = 1$ if $h = \infty$. If L_1, L_2 are two operators with q_1, q_2 respectively on $0 \leq x < a$, and h_1, h_2 are real numbers, there exists a transformation operator $V = V_{\{L_1 L_2 h_1 h_2\}}$ defined for all functions f which are summable on finite subintervals of $0 \leq x < a$ by $Vf(x) = f(x) + \int_0^x K(x, t) f(t)\,dt$ and such that $V\omega_{h_1}^{(1)} = \omega_{h_2}^{(2)}$, where $\omega_{h_1}^{(1)}$, $\omega_{h_2}^{(2)}$ are solutions of $L_1 u + \lambda u = 0$ and $L_2 u + \lambda u = 0$ respectively. The kernel K is real and bounded on each square $0 \leq x \leq b$, $0 \leq t \leq b$, $b < a$. An analogous operator $W = W_{\{L_1 L_2 \theta\}}$ is shown to exist for L_1, L_2 defined on $-a < x < a$ and any real number θ which satisfies $W\omega_h^{(1)} = \omega_{h+\theta}^{(2)}$. The existence of these operators is proved by using classical Fourier methods and Volterra-type integral equations. In particular, no use is made of the partial differential equation method used by Levitan and Povzner. A first application is to the proof of an asymptotic formula for the spectral function ρ associated with a self-adjoint boundary value problem $Lu + \lambda u = 0$, $u(0) \cos \alpha + u'(0) \sin \alpha = 0$ on $0 \leq x < a$ (a boundary condition at a is required in the limit circle case). It is shown that $\rho(\lambda) = 2\pi^{-1}\lambda^{\frac{1}{2}} + o(\lambda^{\frac{1}{2}})$ if $\cot \alpha \neq \infty$, and $\rho(\lambda) = \frac{1}{2}\pi^{-1}\lambda^{\frac{1}{2}} + o(\lambda^{\frac{1}{2}})$ if $\cot \alpha = \infty$. Then it is proved that ρ determines q uniquely. If the problem has only discrete spectrum for two different boundary conditions at zero, then these spectra determine q uniquely. [The last two results were also proved by G. Borg in the paper reviewed above.] Generalizing a result due to Levinson it is next shown that q is uniquely determined by the asymptotic phase provided certain other assumptions are made. A generalized translation operator of the second kind T_x^y, associated with L_1 and L_2, is introduced by

$$T_x^y f = W_{1x} W_{2y} S_x^y (W_1^{-1} f),$$

where

$$W_{1x} f = W_{\{D^2 L \phi_1\}} f(x), \quad W_{2y} f = W_{\{D^2 L \phi_1\}} f(y),$$

$$S_x^y f = \tfrac{1}{2}[f(x+y) + f(x-y)],$$

and $D^2 = d^2/dx^2$. A generalized translation operator of the first kind is defined by $R_x^y f = W_{1x} W_{2y} U_x^y (W_1^{-1} f)$, where $U_x^y f = f(x-y)$. These operators are defined for all f which are summable on each finite real interval. If f is twice differentiable then it can be shown that the function u, $u(x, y; f) = T_x^y f$, is a solution of the partial differential equation $L_{1x} u = L_{2y} u$ satisfying $u(x, 0) = f(x)$, $u_y(x, 0) = \theta_2 f(x)$, and consequently T_x^y in this case reduces to the generalized translation defined by Delsarte. Generalized positive definite functions associated with R_x^y are defined and an analogue of a theorem of Bochner is proved. The problem of the expansion of a function in a Taylor-Delsarte series associated with T_x^y is next investigated, and a necessary and sufficient condition is given which allows such an expansion which is convergent on the whole axis. The properties of the operators $V_{\{D^2 L \theta_1\}}$, $V_{\{D^2 L = \infty\}}$ and the inverses $V_{\{L D^2 \theta\}}$, $V_{\{L D^2 = \infty\}}$ are investigated in detail under the assumption $\int_0^\infty (1+x^2)|q(x)|\,dx < \infty$. An application is given to the determination of the structure of certain commutative Banach algebras (normed rings) associated with L which were introduced by Povzner [Mat. Sbornik N.S. **23** (**65**), 3–52 (1948) = Amer. Math. Soc. Translation no. **5** (1950); these Rev. **10**, 299; **11**, 360] and later studied by Agranovič [Doklady Akad. Nauk SSSR (N.S.) **66**, 1025–1028 (1949); these Rev. **11**, 28]. The last chapter is devoted to the study of generalized almost periodic functions. Let T_x^y be defined as above with $\theta_1 = \theta_2 = 0$. A continuous function f is called (L_1, L_2)-almost periodic if the set $\{T_x^y f\}$, where y is the parameter, is compact in the sense of uniform convergence on the whole real axis. A constructive characterization of (L_1, L_2)-almost periodic functions is given, for the case $\int_0^\infty (1+x^2)|q_j(x)|\,dx < \infty$, $j = 1, 2$, in terms of T_x^y and finite linear combinations of (L_1, L_2)-almost periodic solutions of $L_1 u + \lambda u = 0$.

E. A. Coddington (Los Angeles, Calif.).

Levitan, B. M. Remark on a theorem of V. A. Marčenko. Trudy Moskov. Mat. Obšč. **1**, 421–422 (1952). (Russian)

It is shown how the essential part of the proof of the asymptotic formula for the spectral function given by Marčenko [see the preceding review] follows from a special Tauberian theorem due to Wiener.

E. A. Coddington.

Levitan, B. M. On the asymptotic behavior of the spectral function of a self-adjoint differential equation of the second order. Izvestiya Akad. Nauk SSSR. Ser. Mat. **16**, 325–352 (1952). (Russian)

The asymptotic formula $\rho(\lambda) = 2\pi^{-1}\lambda^{\frac{1}{2}} + o(\lambda^{\frac{1}{2}})$, $\lambda \to +\infty$, obtained by Marčenko [see the second preceding review] for the spectral function is sharpened. Let for $\mu > 0$, $\sigma(\mu) = \rho(\mu^2)$, and define σ on the whole axis to be odd. It is shown that as $\mu \to +\infty$, uniformly in a, $\sigma(\mu + a) - \sigma(a) = 2\pi^{-1}\mu + O(\ln \mu)$, and $\mu^{-1}\int_0^\mu \{\frac{1}{2}[\sigma(\nu + a) + \sigma(\nu - a)] - 2\pi^{-1}\nu\}\,d\nu = O(1)$. If there is no negative spectrum and $h = 0$ or $h = \infty$ (in the latter case ω_h is the solution of $Lu + \lambda u = 0$ satisfying $u(\lambda, 0) = 0$, $u'(\lambda, 0) = \lambda^{\frac{1}{2}}$), and for $x \to 0$, $\int_0^x |q(s)|\,ds = O(x^\alpha)$, where $\alpha > 0$,

then $\sigma(\mu+a)-\sigma(a)=2\pi^{-1}\mu+O(1)$ as $\mu\to+\infty$, uniformly in a. *E. A. Coddington* (Los Angeles, Calif.).

Levitan, B. M. On a special Tauberian theorem. Izvestiya Akad. Nauk SSSR. Ser. Mat. **17**, 269–284 (1953). (Russian)

This is a continuation of the work reviewed above. Using methods similar to those in that paper a special Tauberian theorem is developed, which the author will use, in a later work, to obtain an asymptotic formula for the spectral function for partial differential equations.

E. A. Coddington (Los Angeles, Calif.).

Levitan, B. M. On the asymptotic behavior of the spectral function of a self-adjoint differential equation of the second order and on expansion in eigenfunctions. Izvestiya Akad. Nauk SSSR. Ser. Mat. **17**, 331–364 (1953). (Russian)

Consider the differential equation $y''+(\lambda-q)y=0$, where q is a real function defined on $0\leqq x<\infty$ which is summable on each finite subinterval. Let $\omega_h(\lambda, x)$ denote that solution of this equation satisfying $\omega_h(\lambda, 0)=1$, $\omega_h'(\lambda, 0)=h$. If ρ is any spectral function defined for this equation (and boundary conditions) on $0\leqq x<\infty$ let

$$\theta(x, y, \lambda)=\int_0^\lambda \omega_h(\nu, x)\omega_h(\nu, y)d\rho(\nu)$$

if $\lambda>0$, $\theta(x, y, \lambda)=-\int_0^\lambda \omega_h(\nu, x)\omega_h(\nu, y)d\rho(\nu)$ if $\lambda<0$, and $\theta(x, y, \lambda)=0$ if $\lambda=0$. The function θ is called the spectral function in this paper. The asymptotic nature of θ as a function of λ is investigated, and based on this an equiconvergence theorem with the ordinary Fourier integral expansion is proved. Specifically, if θ^* is the θ for $q\equiv 0$, and q is bounded on each finite interval, and the original boundary-value problem $y''+(\lambda-q)y=0$, $y'(0)-hy(0)=0$, has no negative spectrum, then $\theta-\theta^*$ is bounded for x, y in any finite interval uniformly in λ. If $f\in L^2(0, \infty)$ then $\int_0^\infty[\theta(x, s, \mu^2)-\theta^*(x, s, \mu^2)]f(s)ds$ tends to zero as $\mu\to\infty$, uniformly on every finite x-interval. Analogous results are proved for the case $-\infty<x<\infty$, and the summability of the expansion is discussed. The proofs depend on a special Tauberian theorem and certain estimates obtained in the paper reviewed above. *E. A. Coddington.*

Levitan, B. M. On the spectral function of the equation $y''+\{\lambda-q(x)\}y=0$. Izvestiya Akad. Nauk SSSR. Ser. Mat. **17**, 473–484 (1953). (Russian)

In the notation of the preceding review let

$$\Phi(x, y, \mu)=\theta(x, y, \mu^2)-\theta^*(x, y, \mu^2).$$

It is proved that for fixed $\mu_0>0$, $\Phi(x, y, a+\mu_0)-\Phi(x, y, a)$ tends to zero as $a\to\infty$, uniformly for x, y ranging over a finite region. This generalizes a result due to Marčenko [see the fifth preceding review]. The main tool used is a Tauberian theorem due to Wiener. *E. A. Coddington.*

Levitan, B. M. On expansion in characteristic functions of the equation $y''+\{\lambda-q(x)\}y=0$. Doklady Akad. Nauk SSSR (N.S.) **90**, 17–20 (1953). (Russian)

This is just a statement of the results proved in the two papers reviewed immediately above.

E. A. Coddington (Los Angeles, Calif.).

Kreĭn, M. G. On the transfer function of a one-dimensional boundary problem of the second order. Doklady Akad. Nauk SSSR (N.S.) **88**, 405–408 (1953). (Russian)

Consider the equation and boundary condition

$$y''-q(x)y+\lambda\rho(x)y=0,\quad y'(0)-hy(0)=0.$$

Here q and ρ are measurable functions on $0\leqq x<b$ $(0<b\leqq\infty)$ which are summable on each finite subinterval, $\rho(x)\geqq 0$, and on no subinterval does ρ vanish almost everywhere, h is a real constant, and λ is a complex parameter. Let $\varphi=\varphi(x, \lambda)$ denote the solution of this system satisfying $\varphi(0, \lambda)=1$, $\varphi'(0, \lambda)=h$, and suppose $\sigma(x)=\int_0^x\rho(t)dt$. Let L_σ denote the L^2 space with the measure σ, L_σ^0 the subset consisting of all $f\in L_\sigma$ which vanish on a vicinity of $x=b$. A function τ defined on $-\infty<\lambda<\infty$ $(\tau(\lambda)=\tau(\lambda-0)$, $\tau(0)=0)$ is called a spectral function for the differential system if the mapping $U_\tau: f\to F$, where $f\in L_\sigma^0$, and $F(\lambda)=\int_0^b\varphi(x, \lambda)f(x)d\sigma(x)$, is an isometry of L_σ^0 into L_τ, the L^2 space with measure τ. Such a U_τ can be uniquely extended to an isometry of L_σ into L_τ. If this extension is onto L_τ then τ is said to be orthogonal. If τ is any spectral function then

$$\int_{-\infty}^{\infty}\lambda^{-1}(1-\cos\lambda^{\frac{1}{2}}t)\,\varphi^2(x, \lambda)d\tau(\lambda)<\infty$$

for $0\leqq x<b$, $0\leqq t<2\int_0^x\sqrt{\rho(s)}ds$. The function Φ defined by $\Phi(t)=\int_{-\infty}^{\infty}\lambda^{-1}(1-\cos\lambda^{\frac{1}{2}}t)d\tau(\lambda)$ for $0\leqq t<2\int_0^b\sqrt{\rho(s)}ds$ does not depend on the choice of τ, and is called the transfer function of the differential system. Using this and a result due to Levitan and Melman [same Doklady (N.S.) **81**, 729–731 (1951); these Rev. **13**, 551] if for some spectral function τ with $\int_0^b\sqrt{\rho(s)}ds=\infty$ it is true that

$$\log\left(\int_{-\infty}^{0}\exp\left(|\lambda|^{\frac{1}{2}}t\right)\right)d\tau(\lambda)=O(t^2),\quad t\to\infty,$$

then τ is the unique spectral function of the differential system. Also necessary and sufficient that τ be an orthogonal spectral function is that the linear hull of the functions $\{f_t\}$, where $f_t(\lambda)=\lambda^{-\frac{1}{2}}\sin\lambda^{\frac{1}{2}}t$ $(0<t<\int_0^b\sqrt{\rho(s)}ds)$, be dense in L_τ. In the case where $\rho(x)\equiv 1$ and $h=0$ the function Φ can be thought of as $\Phi(t)=\int_0^t u(\frac{1}{2}s, \frac{1}{2}s)ds$, where u is the solution of the equation $\partial^2u/\partial\xi\partial\eta+q(|\xi-\eta|)=0$ on $0\leqq\xi$, $\eta<b$ satisfying $u(0, \eta)=u(\xi, 0)=1$. An analogous result holds for $h\neq 0$. From this the differentiability properties of Φ are related to those of q. Finally, in order that a given non-decreasing function τ $(\tau(\lambda)=\tau(\lambda-0)$, $\tau(0)=0)$ be a spectral function of some problem with $\rho(x)\equiv 1$ it is necessary and sufficient that Φ have two absolutely continuous derivatives and $\Phi'(0)=1$. Some of these results are extensions of previous work due to Gelfand and Levitan, Marčenko, and Levitan. No proofs are given.

E. A. Coddington (Los Angeles, Calif.).

Kreĭn, M. G. An analogue of the Čebyšev-Markov inequalities in a one-dimensional boundary problem. Doklady Akad. Nauk SSSR (N.S.) **89**, 5–8 (1953). (Russian)

This paper outlines some results obtained by further exploiting the analogy between moment problems and boundary-value problems for second-order linear differential equations. The differential system and notation is that of the previous review. Let, further, ψ denote the solution of the differential equation satisfying $\psi(0, \lambda)=0$, $\psi'(0, \lambda)=1$. Initially the case where φ and ψ are in L_σ for all λ is considered, and an analogue of a theorem of Nevanlinna connected with moment problems is proved. This result characterizes the set of all spectral functions of the differential system, and gives a necessary and sufficient condition that a spectral function be orthogonal. Using this result a set of inequalities is obtained which are the analogue of some due to Čebyšev. An application gives the asymptotic formula for a spectral function obtained by Levitan [see the paper reviewed sixth above]. The method allows various refinements in the remainder estimate under certain additional

assumptions on q. [Reviewer's note: the analogy between boundary-value problems and moment problems has also been considered by H. Weyl, Ann. of Math. (2) 36, 230–254 (1935)]. *E. A. Coddington* (Los Angeles, Calif.).

Yosida, Kôsaku. Correction to my paper "On Titchmarsh-Kodaira's formula concerning Weyl-Stone's eigenfunction expansion" in Nagoya Mathematical Journal, vol. 1 (1950), 49–58. Nagoya Math. J. 6, 187–188 (1953). See same J. 1, 49–58 (1950); these Rev. 13, 39.

Germay, R. H. Sur l'intégrale de Darboux d'une équation aux dérivées partielles du premier ordre de forme résolue et sur sa généralisation. Bull. Soc. Roy. Sci. Liège 22, 264–275 (1953).

By the elimination of constants among the integrated equations of the characteristics, the method of Cauchy defines a general solution of $p_1=f(x_1, \cdots, x_n; p_2, \cdots, p_n, z)$. Among the infinity of possible methods for conducting this elimination the present paper gives the details of one which is different from Darboux's. *J. M. Thomas.*

Mitrinovitch, Dragoslav S. Sur une équation fonctionnelle. C. R. Acad. Sci. Paris 237, 550–551 (1953).

Let $X(x)\neq 0$, $X'=dX/dx$, $Y(y)\neq 0$, $Y'=dY/dy$, etc. Consider the partial differential equations

$$(*) \qquad \frac{r}{X'^2}-\frac{t}{Y'^2}-\left(\frac{X''}{X'^3}p-\frac{Y''}{Y'^3}q\right)=\frac{1}{z}\left(\frac{p^2}{X'^2}-\frac{q^2}{Y'^2}\right),$$

$$(**) \qquad \frac{r}{X'^2}-\frac{t}{Y'^2}-\left(\frac{X''}{X'^3}p-\frac{Y''}{Y'^3}q\right)=0,$$

where $p=\partial z/\partial x, \cdots, t=\partial^2 z/\partial y^2$. Their general solutions are, respectively,

$$z=f(X+Y)g(X-Y) \quad \text{and} \quad z=F(X+Y)+G(X-Y),$$

with f, g, F, G "arbitrary". Their common solutions satisfy $p^2/X'^2-q^2/Y'^2=0$, whose general solutions are of the form $z=\Theta_1(X+Y)$ and $z=\Theta_2(X-Y)$; so for a solution of both (*) and (**), either $g(X-Y)=$constant, $G(X-Y)=$constant, or $f(X+Y)=$constant, $F(X+Y)=$constant. From this it is concluded that the only solutions of the following functional equations are the indicated ones.

$$(1) \qquad f^{(m)}(u)f(v)=f(u)+f^{(n)}(v)$$

[m, n positive integers, superscript denoting derivative]: solution $f(u)\equiv 0$. If $m\neq 0$, $n=0$ in (1), solution is $f=0$. If $m=0$, $n\neq 0$, solutions are $f=0$, $f=1$. If $m=n=0$, solutions are $f=0$, $f=2$. The author does not state, and it is not clear, within what class of functions $\{f(u)\}$ the given solutions of (1) are the only ones. *I. M. Sheffer.*

Aržanyh, I. S. On integration of a canonical system of equations in exact differentials. Uspehi Matem. Nauk (N.S.) 8, no. 3(55), 99–104 (1953). (Russian)

The author is dealing with a generalization of the methods given by Poisson and Jacobi to integrate systems of the type

$$(*) \quad dq_\nu=\sum_{\sigma=1}^{s}\frac{\partial H_\sigma}{\partial p_\nu}dt_\sigma, \quad dp_\nu=-\sum_{\sigma=1}^{s}\frac{\partial H_\sigma}{\partial q_\nu}dt_\sigma \quad (\nu=1, 2, \cdots, n).$$

The following theorems are proved. (1) If

$$\Phi(t_1, \cdots, t_s, q_1, \cdots, q_n, p_1, \cdots, p_n)=A,$$
$$\Psi(t_1, \cdots, t_s, q_1, \cdots, q_n, p_1, \cdots, p_n)=B$$

are integrals of a completely integrable system (*) the constant Poisson-bracket $(\Phi\Psi)=C$ is again an integral of this system. (2) If $V(t_1, \cdots, t_s, q_1, \cdots, q_n, a_1, \cdots, a_n)$ is a complete integral of the partial differential system

$$\frac{\partial V}{\partial t_\sigma}+H_\sigma\left(t_1, \cdots, t_s, q_1, \cdots, q_n, \frac{\partial V}{\partial q_1}, \cdots, \frac{\partial V}{\partial q_n}\right)=0, \qquad \sigma=1, 2, \cdots, s,$$

the solution of the system (*) associated with this partial system is given by

$$p_\nu=\frac{\partial V}{\partial q_\nu}, \quad \frac{\partial V}{\partial a_\nu}=b_\nu \quad (\nu=1, 2, \cdots, n)$$

(as a generalization of Jacobi's theorem). Further conclusions follow in the case that the expressions H_σ are independent of the parameters $t_1, \cdots, t_s$. Finally an example is given. *M. Pinl* (Dacca).

Hukuhara, Masuo. On the regular solutions about a singular point of a linear partial differential equation of the first order. Mem. Fac. Sci. Kyūsyū Univ. A. 5, 55–60 (1950). (Japanese)

Let the coefficients A_j $(j=1, \cdots, n)$ and A of an equation $\sum_{j=1}^{n}A_j\partial f/\partial x_j=A$ be given functions of $(x_1, \cdots, x_n)$ vanishing at $(0, \cdots, 0)$ and regular in its neighborhood. If the roots λ_j of its indicial equation at the singularity $(0, \cdots, 0)$ satisfy $\Re\lambda_j e^{i\omega}<0$ $(j=1, \cdots, n)$, ω being a real constant, then there exists a regular solution vanishing at $(0, \cdots, 0)$. The author proposes a method for estimating a range of existence of the solution and its bounds.

Let a solution of

$$dx_j/dt=A_j \quad (j=1, \cdots, n)$$

be $x_j=P_j(y_1, \cdots, y_n)$ with $P_j(0, \cdots, 0)=0$, where the P's are regular at $(0, \cdots, 0)$ and $y_j=\varphi_j(t, C_1, \cdots, C_j)\exp\lambda_j t$, the C's being constants and φ_j being a polynomial with respect to t, $C_1, \cdots, C_j$ such that φ_j-C_j contains t as a factor and does not depend on C_j. Denote by L_t the half-line with inclination ω starting at t. Then, the function defined by

$$(*) \qquad f=-\int_{L_t}A(P_1, \cdots, P_n)dt$$

depends on $x_1, \cdots, x_n$ alone and represents the regular solution vanishing at $(0, \cdots, 0)$ of the original equation. Let D and Δ be domains containing the origins in $2n$-dimensional x- and y-spaces, respectively, such that Δ is mapped by $x_j=P_j(y_1, \cdots, y_n)$ $(j=1, \cdots, n)$ one-to-one onto D. Suppose that $A(P_1, \cdots, P_n)$ is regular in Δ and that $(C_1, \cdots, C_n)\in\Delta$, $t\in L_0$ implies $(y_1, \cdots, y_n)\in\Delta$. It is then concluded that $f(x_1, \cdots, x_n)$ is regular in D. An estimation of bounds may be carried out by means of (*). *Y. Komatu* (Tokyo).

Lions, Jacques-Louis. Problèmes aux limites. I, II, III. C. R. Acad. Sci. Paris 236, 2373–2375, 2470–2472; 237, 12–14 (1953).

These notes announce results on the classical problems (Dirichlet, Neumann, Cauchy, etc.) for linear second-order partial differential equations when these problems are stated appropriately in the context of L. Schwartz' Théorie des distributions [t. I, II, Hermann, Paris, 1950, 1951; these Rev. 12, 31, 833]. The typical conclusion is that the problem has a unique solution and that this solution depends continuously on the data of the problem. The first note discusses elliptic problems using the L^2 norms of the unknown and its first and second derivatives. In the second

note, also devoted to elliptic problems, the dimension is $\geqq 3$, certain L^p norms are used, and the result is more complicated; in a special case a consistency condition is required and uniqueness can be guaranteed only up to an additive constant. For the special regions used in the second note (ouverts de Soboleff) estimates are stated for L^p norms in terms of L^2 norms of first derivatives; if the region is of finite measure the estimate reduces, for $p=2$, to Poincaré's inequality. For hyperbolic or parabolic problems, to which the third note is devoted, the solution may be represented explicitly as a convolution of the nonhomogeneous term of the differential equation with a distribution found by means of a Laplace transformation with respect to the time-variable. Numerous extensions and generalizations are indicated. *F. A. Ficken* (Knoxville, Tenn.).

***Smirnov, M. M. Zadači po uravneniyam matematičeskoĭ fiziki. [Problems in the equations of mathematical physics.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 72 pp. .90 rubles.

Problems: Reduction to canonical form of partial differential equations in the case of two independent variables; The method of characteristics; The method of separation of variables. Answers and hints.

Hartman, Philip, and Wintner, Aurel. On the local behavior of solutions of non-parabolic partial differential equations. Amer. J. Math. **75**, 449–476 (1953).

This paper is concerned with the local behavior of the solutions of elliptic partial differential equations in two independent variables. Its main results can be formulated in two theorems as follows:

Theorem A. In a circle K: $x^2+y^2<R^2$, suppose $a(x, y)$, $b(x, y)$, $c(x, y)$ to be functions of class C^1 which are subject to the restriction $ac-b^2<0$ and to the normalizing conditions $a(0, 0)=c(0, 0)=1$, $b(0, 0)=0$; suppose $d(x, y)$, $e(x, y)$, $f(x, y)$ to be continuous functions in K. Then if $u(x, y)$ is a function of class C^2 in K which satisfies the elliptic, homogeneous, linear, partial differential equation

$$(*) \qquad au_{xx}+2bu_{xy}+cu_{yy}+du_x+eu_y+fu=0,$$

the following statements are true: (1) If the condition $(**)$ $u(x, y)=o(\rho^n)$ as $\rho=(x^2+y^2)^{1/2}\to 0$ is satisfied for some non-negative integer n, then $\lim_{\rho\to 0}(u_y+iu_x)/(x+iy)^n$ exists. (2) Unless $u\equiv 0$, there exists a least integer m such that $(**)$ fails to hold for $n=m$. (3) If d, e, f are of class C^1 in K, and if $(**)$ is valid for a particular positive integer n, then both limits

$$\lim_{\rho\to 0}(u_{xy}+iu_{xx})/(x+iy)^{n-1}, \quad \lim_{\rho\to 0}(u_{yy}+iu_{xy})/(x+iy)^{n-1}$$

exist for this n.

Theorem B. Let $A=A(x, y)$ and $B=B(x, y)$ be n by n matrix functions of class C^1 and of class C^0, respectively, on the closure of the (x, y)-domain D_R: $x>0$, $x^2+y^2<R^2$, and let the elementary divisors of A be simple. Let $u=u(x, y)$ be any vector function, with n components, which is of class C^1 and satisfies $u_x+Au_y=Bu$ on D_R, is continuous on the closure of D_R, and which vanishes for $x=0$, $|y|<R$. Then there exists a positive ϵ $(<R)$ such that $u(x, y)\equiv 0$ on D_ϵ.

These theorems require less differentiability of coefficients than do otherwise comparable results of T. Carleman [C. R. Acad. Sci. Paris **197**, 471–474 (1933); Ark. Mat. Astr. Fys. **26B**, no. 17 (1939)], L. Bers [Proc. Nat. Acad. Sci. U. S. A. **36**, 130–136 (1950); **37**, 42–47 (1951); Theory of pseudo-analytic functions, New York Univ., 1953; these Rev. **12**, 173; **13**, 352; **15**, 211], and of I. N. Vekua [Mat. Sbornik N.S. **31(73)**, 217–314 (1952); these Rev. **15**, 230]. Both the theorems are extended to non-linear equations and are also generalized in other ways. *A. Douglis*

***Kupradze, V. D. Graničnye zadači teorii kolebaniĭ i integral'nye uravneniya. [Boundary problems of the theory of vibrations and integral equations.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Msocow-Leningrad, 1950. 280 pp.

The present book is intended to survey, in monograph form, the results obtained by the author in the disciplines mentioned in the title, and hence does not contain an account of the contributions of other writers. There is a brief informative introduction containing historical remarks and a synopsis of the work. The main concern is with the boundary-value problems relating to the equation

$$(*) \quad \Delta^*\vec{u}+k_2^2\vec{u}=0; \quad \Delta^*\equiv\Delta+\frac{\lambda+\mu}{\mu}\,\mathrm{grad\ div}, \quad k_2^2=\sigma\frac{\omega^2}{\mu},$$

where $\vec{u}$ (u_1, u_2, u_3) is the displacement vector, λ and μ are elastic constants, σ is the density, and ω the frequency of vibration. (If $\omega=0$, one obtains the equation of static elasticity $\Delta^*\vec{u}=0$; if $\omega\neq 0$, but $\lambda=-\mu\neq 0$, one obtains as a special case the equation of a vibrating membrane $\Delta u+k^2u=0$; while if $\omega=0$, $\lambda=-\mu\neq 0$, one obtains as a special case Laplace's equation $\Delta u=0$.) Boundary conditions are of the general form (subscript i refers to the interior and subscript a to the exterior problem) $\alpha\vec{u}_{i,a}+\beta T_{i,a}=\vec{f}$, on a closed surface S, smooth in the sense of Liapounoff, where T denotes the stress vector corresponding to the unit normal to S.

Chapter 1 contains a discussion of the general properties of the equation of a vibrating membrane and chapter 2 contains a solution of the fundamental exterior boundary-value problems. Chapter 3 is concerned with boundary-value problems in the theory of electromagnetic vibrations. Chapter 4 deals with vibrations of an elastic medium, and presents a generalization of the classical theory of the potential in order to construct the solution of the appropriate boundary-value problems (certain fundamental solutions are introduced and used to form what are called potentials of simple, double, and antenna layers over S, which are used to obtain Fredholm integral equations equivalent to the boundary-value problems [see V. D. Kupradze, Uspehi Matem. Nauk (N.S.) **5**, no. 3(37), 190–193 (1950); these Rev. **12**, 263]). The following uniqueness theorem for the exterior boundary-value problem is given: A solution $\vec{u}$ of equation $(*)$ which is regular on the exterior of S and satisfies on S the boundary condition $\alpha\vec{u}_a+\beta T_a=0$ is identically zero. By saying that $\vec{u}$ is regular is meant that $\vec{u}$ has continuous first partial derivatives on $S+\mathrm{ext}\ S$, continuous second partial derivatives on ext S, and $\vec{u}$ satisfies the radiation condition at infinity:

$$\lim_{r\to\infty}\vec{u}^{(j)}=0; \quad \lim_{r\to\infty} r\left(\frac{\partial\vec{u}^{(j)}}{\partial r}-ik_j\vec{u}^{(j)}\right)=0; \quad k_1^2=\sigma\frac{\omega^2}{\lambda+2\mu}, \quad j=1, 2;$$

where $u^{(1)}$ and $u^{(2)}$ are longitudinal and transverse components of $\vec{u}$: $\vec{u}=\vec{u}^{(1)}+\vec{u}^{(2)}$; $\Delta\vec{u}^{(1)}+k_1^2\vec{u}^{(1)}=0$, rot $\vec{u}^{(1)}=0$; $\Delta\vec{u}^{(2)}+k_2^2\vec{u}^{(2)}=0$, div $\vec{u}^{(2)}=0$. The last chapter is devoted to integral equations with singular kernels, with particular attention to the development of explicit formulas for the solutions. These considerations stem from one of the two methods of approach to the subject which were used by

T. Carleman [Ark. Mat. Astr. Fys. 16, no. 26 (1922)]; the other approach having been considered mainly by N. I. Mushelišvili [Singular integral equations, OGIZ Moscow-Leningrad, 1946; these Rev. 8, 586] and S. G. Mihlin [Uspehi Matem. Nauk (N.S.) 3, no. 3(25), 29–112 (1948); these Rev. 10, 305]. There is a simple direct proof of the Poincaré-Bertrand formula for the inversion of the order of integration of iterated singular integrals [V. D. Kupradze, Trudy Tbiliss. Gos. Univ. im. Stalin. 23, 159–164 (1942); F. Tricomi, Math. Z. 27, 87–133 (1927)]. *J. B. Diaz.*

Zyuz'ko, M. P. On spectral properties of the operator $-\Delta u+cu$ in an unbounded space of an arbitrary number of dimensions. Doklady Akad. Nauk SSSR (N.S.) 90, 957–959 (1953). (Russian)

The main results of Povzner [Mat. Sbornik N.S. 32(74), 109–156 (1952); these Rev. 14, 755] concerning the resolvent and the spectral resolution of the Schrödinger operator are extended to dimension $n>3$. *L. Gårding.*

Glazman, I. M. On the character of the spectrum of multidimensional singular boundary problems. Doklady Akad. Nauk SSSR (N.S.) 87, 171–174 (1952). (Russian)

This note continues an earlier one by the same author [same Doklady (N.S.) 87, 5–8 (1952); these Rev. 14, 1088]. Consider the Schrödinger operator $l=-\Delta+q$ in n-dimensional space E. The function q is supposed to be real and continuous. Let L be the closure in $L^2(E)\times L^2(E)$ of l defined on all twice continuously differentiable functions vanishing outside compact sets. L is symmetric and has (in general many) self-adjoint extensions $\bar{L}$. Let $C(L)$ be the intersection of the continuous spectra $C(\bar{L})$ of all $\bar{L}$. An open subset Ω of E is called quasi-conical if it contains arbitrarily large spheres, quasi-cylindrical if it contains an infinity of non-intersecting congruent spheres and quasi-bounded if it is not quasi-cylindrical. Let Ω_η be the part of Ω where $|q|>\eta$. It is shown that $C(L)$ contains the interval $[0,\infty)$ provided that $\int_{\Omega_\eta}|q|^2dt<\infty$ for all η and some quasi-conical Ω. A special case is $\Omega=E$ and $\lim_{x\to\infty}q(x)=0$. It is also shown that if for some quasi-conical Ω,

$$\omega=\limsup_{\Omega\ni x\to\infty}q(x)-\liminf_{\Omega\ni x\to\infty}q(x)<\infty$$

then $[\lambda,\lambda+\omega]\cap C(L)$ is not empty for any $\lambda>0$, and if $\lim_{\Omega\ni x\to\infty}q(x)=0$ for some quasi-cylindrical Ω then $C(L)\cap[N,\infty)$ is not empty for any N. If $\lim_{x\to\infty}q(x)=q_0$ then $(-\infty,q_0)\cap C(L)$ is empty and if $\lim_{x\to\infty}q(x)=-0$ and $\gamma_k\int_{\gamma_k}^{\infty}\sup_{|x|=r}|q(x)|dr>1$ for some sequence $\gamma_k\to\infty$, then $\lambda=0$ is a limit point for the negative eigenvalues of L (in both cases L itself is self-adjoint).

For the self-adjoint operator $\bar{L}$ which equals the closure in $L^2(\Omega)\times L^2(\Omega)$ of $-\Delta$ defined on all twice continuously differentiable functions vanishing outside compact subsets of Ω it is shown that $[0,\infty)\subset C(L)$ if Ω is quasi-conical and that $C(\bar{L})$ extends to infinity if Ω is quasi-cylindrical (Molčanov [ibid. 83, 17–18 (1952); Trudy Moskov. Mat. Obšč. 2, 169–199 (1953); these Rev. 14, 473; 15, 224] has shown that $C(\bar{L})$ is empty if Ω is quasi-bounded).

L. Gårding (Princeton, N. J.).

Jones, D. S. The eigenvalues of $\nabla^2u+\lambda u=0$ when the boundary conditions are given on semi-infinite domains. Proc. Cambridge Philos. Soc. 49, 668–684 (1953).

The spectrum of the Laplace operator $-\Delta$ associated with an open bounded subset T of real n-dimensional space and functions satisfying one of the classical boundary conditions $u=0$ or $u_\nu+\sigma u=0$ (u_ν is the derivative along the interior normal and σ is real) is investigated when T has the shape of a cone or a semi-infinite cylinder. The number of eigenvalues in the lower discrete part of the spectrum is estimated in terms of the spectrum belonging to a piece cut off from T. When the boundary condition is $u=0$ and the spectrum is not discrete above m, then it will still be not discrete above m if T increases (the boundary condition may also change). If T contains a cylinder whose continuous spectrum with respect to the boundary condition $u=0$ contains an interval $[m,\infty)$, then the spectrum of T is continuous in the same interval with the possible exception of a discrete nowhere-dense point-spectrum with finite multiplicities. If T contains a cone, the continuous spectrum contains the interval $[0,\infty)$ and if T itself is conical at infinity and the boundary condition there is $u=0$ or $u_\nu=0$, then there are no discrete eigenvalues in the same interval. (Partly more general results in the same direction have been obtained by Glazman in the paper reviewed above; cf. also Rellich [Courant Anniversary Volume, Interscience, New York, 1948, pp. 329–344; these Rev. 9, 355] and Molčanov [Doklady Akad. Nauk SSSR (N.S.) 83, 17–18 (1952); Trudy Moskov. Mat. Obšč. 2, 169–199 (1953); these Rev. 14, 473; 15, 224)]. Extensions to the operator $-\Delta+b$ are given together with several other results. It is shown that a solution of $\Delta u+\lambda u=0$ ($\lambda>0$), in a region T conical at infinity which satisfies $u=0$ or $u_\nu\neq0$ at the boundary and a radiation condition at infinity vanishes in T. The results are applied to the theory of surface waves. There may exist a negative point-spectrum for a canal of finite width whose bed carries a protrusion over a finite length but is otherwise of uniform depth. *L. Gårding* (Princeton, N. J.).

Pini, Bruno. Sulle singolarità delle soluzioni della equazione $\Delta u+cu=0$. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 14, 21–26 (1953).

L'auteur perfectionne davantage au sujet de l'équation $\Delta u+cu=0$ dans l'espace R^n, un résultat d'unicité de Picone, qu'il avait déjà amélioré récemment [Boll. Un. Mat. Ital. (3) 8, 19–25 (1953); ces Rev. 15, 221]. Il s'agit de voir que, avec une certaine borne supérieure de c, toute solution est nulle dès qu'elle s'annule à la frontière, sauf sur certaines parties A_i où sa croissance est convenablement limitée. Ces parties A_i au lieu d'être des points isolés connue chez Picone peuvent être des variétés de dimension $\leqq n-2$ et les limitations de croissance sont améliorées par des conditions en moyenne faisant intervenir, au lieu d'une fonction de la distance à A_i (comme la puissance $-\alpha$, $\alpha>0$) un potentiel d'ordre α de la mesure sur cette variété. *M. Brelot.*

Ėĭdus, D. M. Estimates of the modulus of characteristic functions. Doklady Akad. Nauk SSSR (N.S.) 90, 973–974 (1953). (Russian)

Let $u_1, u_2, \cdots$ and $\lambda_1, \lambda_2, \cdots$ be the normalized eigenfunctions and the corresponding eigenvalues of a m-dimensional vibrating membrane fixed at its boundary, which is supposed to be smooth. Applying Green's formula to u_n^2 and an elementary solution of $\Delta+\lambda_n$, the author proves that $|u_n(x)|\leqq c\lambda_n^{m/4}$, where c is a constant and x ranges over the region Ω covered by the membrane. On compact subsets of Ω the inequality is valid with $\frac{1}{4}m$ replaced by $\frac{1}{4}(m-1)$ and this result is the best possible. The note improves a result by Smolickiĭ [same Doklady (N.S.) 74, 205–208 (1950); these Rev. 12, 411]. *L. Gårding.*

Browder, Félix. Le problème des vibrations pour un opérateur aux dérivées partielles self-adjoint et du type elliptique, à coefficients variables. C. R. Acad. Sci. Paris **236**, 2140–2142 (1953).

Let K be an elliptic operator,

$$Ku(x)=(-1)^m\sum_{|\nu|\leq 2m}a_\nu(x)D^\nu u(x)$$

$$(D^\nu=\partial^{|\nu|}/\partial x_1^{\nu_1}\cdots\partial x_n^{\nu_n},\ |\nu|=\nu_1+\cdots+\nu_n),$$

with sufficiently regular coefficients defined in an open and bounded subset D of real n-space. It is assumed that the polynomial $a(x,\xi)=\sum_{|\nu|=2m}a_\nu(x)\xi_1^{\nu_1}\cdots\xi_n^{\nu_n}$ is uniformly positive definite in D and that K is formally self-adjoint. Let $H_m(D)$ be the closure with respect to the square norm $\int_D\sum_{|\nu|\leq m}|D^\nu f|^2dx$ of the set of all infinitely differentiable functions with compact supports in D. It is known that there exists a complete set of eigenfunctions $\varphi_1,\varphi_2,\cdots$ of K in $H_m(D)$, orthonormalized with respect to the scalar product $\int_D f\bar{g}dx$. Let $\lambda_1,\lambda_2,\cdots$ be the corresponding eigenvalues. If $2m>n$ and t is large enough, then Green's function $g_t(x,y)=\sum\overline{\varphi_i(x)}\varphi_i(y)(\lambda_i+t)^{-1}$ exists and is continuous and bounded on $D\times D$. Using a method of the reviewer [Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplementaire, 109–118 (1952); these Rev. **14**, 878], the author proves that

$$g_t(x,y)=t^{\nu-1}\left(\delta_{xy}(2\pi)^{-n}\int(a(x,\xi)+1)^{-1}d\xi+o(1)\right),$$

($t\to\infty$, $\nu=n/2m$, $\delta_{xy}=0$ when $x\neq y$ and $=1$ otherwise) and that $g_t=t^{\nu-1}O(1)$ uniformly on $D\times D$. Following Carleman [Attonde Skandinaviska Matematikerkongressen, Stockholm, 1934, Ohlsson, Lund, 1935, pp. 34–44], he then deduces the asymptotic formulas

$$\sum_{\lambda_i\leq t}1=t^{n/2m}\left\{(2\pi)^{-n}\int_D\rho(x)dx+o(1)\right\}$$

and

$$\sum_{\lambda_i\leq t}\overline{\varphi_i(x)}\varphi_i(y)=t^{n/2m}\left\{(2\pi)^{-n}\delta_{xy}\rho(x)+o(1)\right\}$$

from a suitable Tauberian theorem ($\rho(x)=\int_{a(x,\xi)<1}d\xi$).

L. Gårding (Princeton, N. J.).

Položil, G. N. The theorem on preservation of domain for certain elliptic systems of differential equations and its applications. Mat. Sbornik N.S. **32** (74), 485–492 (1953).

Let $u(x,y)$, $v(x,y)$ be continuously differentiable functions satisfying the linear elliptic system

$$au_x+bu_y-v_y=0,\quad du_x+cu_y+v_x=0,$$

where the coefficients a, b, c, d have continuous derivatives satisfying a Hölder condition. Set $w(z)=u+iv$. The author's result may be stated as follows. Theorem: If $w(z)\not\equiv$const., then the mapping $w=w(z)$ is interior in the sense of Stoïlov, and the Jacobian $u_xv_y-u_yv_x$ does not vanish at points at which the mapping is one-to-one. The proof is based on the Lemma: Let $\omega=\omega(z)$ be a continuously differentiable homeomorphism of $|z|<1$ onto $|\omega|<1$, $\omega(0)=0$. Assume that the Jacobian of the mapping is positive, the dilation coefficient is Hölder continuous at the origin, and equals 1 at the origin. Then the ratio $|\omega/z|$ is bounded and bounded away from zero.

In 1947 the author proved his theorem for the special case $a=c$, $b=d=0$ [Doklady Akad. Nauk SSSR (N.S.) **58**, 1275–1278 (1947); these Rev. **9**, 507]. He cites a 1946 abstract by the reviewer and Gelbart [Bull. Amer. Math. Soc. **52**, 64 (1946)], but seems to be unaware of the fact that his present theorem is contained in the reviewer's theory of pseudo-analytic functions [Proc. Nat. Acad. Sci. U. S. A. **36**, 130–136 (1950); **37**, 42–47 (1951); these Rev. **12**, 173; **13**, 352]. [As a matter of fact, the theorem remains true if the coefficients are non-differentiable but satisfy a Hölder condition, as will be shown in a forthcoming paper.]

L. Bers (New York, N. Y.).

Nitsche, Joachim, und Nitsche, Johannes. Allgemeine Randwertprobleme für Systeme elliptischer Differentialgleichungen; die Zurückführung auf eine von F. Noether untersuchte Klasse singulärer Integralgleichungen. Rend. Circ. Mat. Palermo (2) **2**, 40–45 (1953).

The authors consider the elliptic partial differential equation (1) $\Delta\phi+p\phi_x+q\phi_y+r\phi=0$; solutions $\phi(x,y)$ are sought, twice differentiable in a planar domain T, continuously differentiable on $\bar{T}=T+S$ and satisfying

(2) $\quad -a(s)\partial\phi/\partial n+b(s)\partial\phi/\partial s+c(s)\phi=\pi f(s)$ on S;

$a,\cdots,f$ are continuous as functions of arc s. As to the coefficients p, q, r, one may assume, for instance, that they possess first order partials of Hölder class. By using a fundamental solution of (1) [Lichtenstein, Encykl. Math. Wiss., Bd II3, Heft 8, Teubner, Leipzig, 1924, pp. 1277–1334] vanishing on S, the problem is reduced to a singular integral equation of F. Noether type [Math. Ann. **82**, 42–63 (1920)]. If the index of the boundary condition (2) is $n>0$, then the homogeneous problem [(1), (2)] has at least $2n$ linearly independent solutions; if $n<0$, there exists a solution of the nonhomogeneous problem [(1), (2)] when f satisfies certain p ($\geq 2|n|$) integral conditions. Under hypotheses of the above description the authors make a similar study of the elliptic system $U_x-V_y=AU+BV$, $U_y+V_x=\bar{A}U+\bar{B}V$, in which case the domain has to be suitably restricted.

W. J. Trjitzinsky (Urbana, Ill.).

Tricomi, Francesco G. Un teorema di media per certe equazioni di tipo ellittico. Rend. Sem. Mat. Univ. Padova **22**, 350–353 (1953).

The elliptic partial differential equations studied are of the form (1) $\Delta^2u+az_x+bz_y+c=f$, where Δ^2 stands for the Laplacian in two variables x, y, and a, b, c, and f are functions of x, y. The fundamental solution of the equation adjoint to (1) is denoted by $U=U(x,y;\xi,\eta)$; it has a standard logarithmic singularity at (ξ,η). Let (ξ,η) be any point, γ a level curve $U=C$ of $U(x,y;\xi,\eta)$ containing (ξ,η), D the region bounded by γ, and z a solution of (1) regular in D. The author proves under appropriate smoothness assumptions the following formula of mean-value type:

$$2\pi z(\xi,\eta)=\oint_\gamma z\frac{\partial U}{\partial n}ds-C\iint_D(c-a_x-b_y)zdxdy$$

$$+\iint_D(C-U)fdxdy.$$

The proof is very simply carried out by using Green's formula, and the result may be used, for example, in proving maximum modulus theorems for solutions of elliptic equations.

J. W. Green (Los Angeles, Calif.).

Keller, Joseph B. The scope of the image method. Comm. Pure Appl. Math. 6, 505–512 (1953).
Reviewed earlier as a report [cf. these Rev. 14, 877].

Smith, J. J. Tables of Green's functions, Fourier series, and impulse functions for rectangular co-ordinate systems. Trans. Amer. Inst. Elec. Engrs. 70, 22–30 (1951).

The paper gives a collection of Green's function for the solution of boundary-value problems in the Laplace and Poisson equations, the heat equation, and the wave equation, in one, two, and three dimensional rectangular regions. On the boundaries any linear combination, with constant coefficients, of the unknown function and its normal derivative is to be prescribed. Unbounded regions are included. Green's functions are listed as infinite trigonometric series or integrals, generally nonconvergent. Relations between those series and orthogonal series representations of functions and of impulse functions are tabulated. Explanations of the complex notation used here are not easy to find.

R. V. Churchill (Ann Arbor, Mich.).

Ważewski, T. Sur une relation entre la façon de la mise en équation du problème physique et la notion des solutions généralisées des équations aux dérivées partielles du second ordre. Bull. Acad. Polon. Sci. Cl. III. 1, 79–82 (1953).

The initial-value problem for the vibrating string usually demands a solution of the wave equation having continuous second derivatives. The problem is formulated here so as to demand a solution with continuous first derivatives of a certain integro-differential equation, and the solutions are identified with Sobolev's "generalized solutions" of the wave equation. The physical significance of these results is discussed very briefly. *F. A. Ficken* (Knoxville, Tenn.).

Mullin, Charles J. Solution of the wave equation near an extremum of the potential. Physical Rev. (2) 92, 1323–1324 (1953).

Kay, Irvin. The diffraction of an arbitrary pulse by a wedge. Comm. Pure Appl. Math. 6, 419–434 (1953).

A pulse wave is defined as a moving discontinuity surface behind which there is a prescribed field which satisfies the wave equation $\Delta u = u_{tt}/c^2$ and in front of which the field is zero. A diffraction problem arises when an obstacle exists in the path of the moving pulse. Since the wave equation is of hyperbolic type, it is possible to make use of the fact that the solution in some regions of the space-time continuum is entirely independent of boundary or initial conditions imposed in certain other regions; thus it is possible here to break up a complicated problem into several simple parts, a simplification which does not arise in "monochromatic" diffraction problems since the equation is then the elliptic equation $\Delta u + k^2 u = 0$.

In the present paper is considered the two-dimensional problem in which an arbitrary pulse wave is incident on an infinite wedge. If the pulse is incident on the edge of the wedge at the instant $t=0$, the field outside and on the light-cone $(x^2+y^2)^{1/2}=ct$ is known by the remark made above. The problem is thus reduced to that of determining a solution of $u_{xx}+u_{yy}=u_{tt}/c^2$ which takes given values on the light-cone and vanishes on the faces of the wedge.

The interesting change of variable:

$$r=(x^2+y^2)^{1/2}, \quad \phi=(x^2+y^2)^{1/2}-ct, \quad \theta=\tan^{-1} y/x$$

reduces the equation to

$$u_{\phi\phi}+r^2u_{rr}+ru_r+2r^2u_{\phi r}+ru_\phi=0.$$

This has to be solved in the region $-\infty \leqq \phi \leqq 0$, $0 \leqq r \leqq \infty$, $\beta \leqq \theta \leqq 2\pi-\beta$ and satisfy boundary conditions on $\phi=0$, $\theta=\beta$, $\theta=2\pi-\beta$.

By separation of variables, $u=R(r)\Theta(\theta)\Phi(\phi)$, particular solutions

$$e^{\lambda(r-\phi)}I_k(\lambda r)\sin k(\theta-\beta)$$

are obtained. The separation constant λ is capable of continuous variation, but k can take only the values

$$k_n=n\pi/2(\pi-\beta) \quad (n=1, 2, 3, \cdots)$$

since u vanishes on $\theta=\beta$ or $2\pi-\beta$. The formal solution is thus

$$u=\sum_1^\infty \sin k_n(\theta-\beta)\int_{L_n} A_n(\lambda)e^{\lambda(r-\phi)}I_{k_n}(\lambda r)d\lambda$$

where the contours L_n and the functions $A_n(\lambda)$ have to be determined to satisfy the conditions on the light-cone $\phi=0$. There is no space here to discuss the rather difficult determination of the $A_n(\lambda)$ by means of the Mellin inversion formula. *E. T. Copson* (St. Andrews).

Szarski, J., et Ważewski, T. Sur une méthode de comparaison des équations hyperboliques aux dérivées partielles du second ordre avec les équations différentielles ordinaires. Bull. Acad. Polon. Sci. Cl. III. 1, 6–10 (1953).

Let $H(u)=\sum_{i,k=1}^m a_{ik}(x_1, \cdots, x_m)\partial^2 u/\partial x_i \partial x_k$ be a totally hyperbolic operator with x_m the distinguished direction, and consider the differential inequality

$$|H(u)| \leqq \sum_{j=1}^m |b_j(x_1, \cdots, x_m)|\left|\frac{\partial u}{\partial x_j}\right| + |c(x_1, \cdots, x_m)|\cdot|u| + |f(x_1, \cdots, x_m)|.$$

For functions u satisfying such an inequality the authors state an estimate for the integral

$$\int\left(\sum_{i=1}^m (\partial u/\partial x_i)^2+u^2\right)dx_1\cdots dx_{m-1}$$

where the range of integration is in a portion of the hyperplane $x_m=\text{const}$. With the aid of this estimate a uniqueness theorem and the continuous dependence on initial values are stated for solutions of the hyperbolic equation $H(u)=f(x_1, \cdots, x_m, u, \partial u/\partial x_1, \cdots, \partial u/\partial x_m)$. The estimate for the above integral generalizes similar inequalities obtained previously by Friedrichs and Lewy [Math. Ann. 98, 192–204 (1927)] and Sobolev [The equations of mathematical physics, 2nd ed., Gostehizdat, Moscow-Leningrad, 1950, p. 303; these Rev. 13, 42]. *M. H. Protter.*

Diaz, J. B., and Weinberger, H. F. A solution of the singular initial value problem for the Euler-Poisson-Darboux equation. Proc. Amer. Math. Soc. 4, 703–715 (1953).

The present paper contains a new solution of the Cauchy problem

$$(*) \qquad \Delta u = u_{tt}+\frac{k}{t}u_t; \quad \Delta=\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_m^2}; \quad t>0;$$

$$u(x_1, \cdots, x_m, 0)=f(x_1, \cdots, x_m), \quad u_t(x_1, \cdots, x_m, 0)=0,$$

k being a real or complex parameter. The solution is found for the exceptional values $k=-1, -3, \cdots$, even in the case of arbitrary functions $f(x_i)$. Its t derivative of order $1-k$

is logarithmic at $t=0$ when $f(x_i)$ is not polyharmonic of order $(1-k)/2$. Hadamard's "method of descent" can be used to obtain solutions of (*) for $k=m, m+1, m+2, \cdots$ from the known solution for $k=m-1$. The resulting formula gives a solution of the problem for any k with $\operatorname{Re} k>m-1$. In this part of his theory the author is in common with Weinstein's procedure [cf. A. Weinstein, Proceedings of the Fifth Symposium in Applied Mathematics, 1952 (in print)]. For $k\leqq m-1$ the definite integral, in terms of which the solution is expressed for $\operatorname{Re} k>m-1$, is divergent. The method used by the author differs from that used by M. Riesz [Acta Math. 81, 1–223 (1949); these Rev. 10, 713] for solving regular Cauchy problems. Both methods employ analytic continuation of definite integrals. But neither a fundamental solution nor a Green's identity is used in this paper. A special case leads to a generalization of Huygen's principle. *M. Pinl* (Dacca).

Smirnov, M. M. Functionally invariant solutions of equations of hyperbolic-parabolic type with three independent variables. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 509–512 (1953). (Russian)

A solution $f(x, y, t)$ of the equation

$$Lu\equiv a_{11}u_{xx}+2a_{12}u_{xy}+a_{22}u_{yy}-u_{tt}+b_1u_x+b_2u_y-\beta u_t=0$$

where the coefficients are functions of the independent variables is said to be functionally invariant if $G(f)$ is a solution for arbitrary (smooth) G. For the case where $a_{11}a_{22}-a^2_{12}=0$ the author gives conditions on the coefficients in order that Lu should possess functionally invariant solutions. The existence of such solutions is employed to solve the Cauchy problem for the case $b_1=b_2=\beta=0$, a_{11}, a_{12}, a_{22} constant. *M. H. Protter* (Berkeley, Calif.).

Pini, Bruno. Sui sistemi di equazioni lineari a derivate parziali del secondo ordine dei tipi ellittico e parabolico. Rend. Sem. Mat. Univ. Padova 22, 265–280 (1953).

Pini, Bruno. Osservazioni sulle soluzioni dei sistemi di equazioni a derivate parziali lineari di tipo ellittico. Rend. Sem. Mat. Univ. Padova 22, 366–379 (1953).

Let D be a domain in the (x, y)-plane whose boundary Γ is a simple closed curve of class two. Let $\Gamma(\alpha)$ be a one parameter family of curves interior to D and parallel to Γ, such that as $\alpha\to 0$ the curves $\Gamma(\alpha)$ approach the boundary curve Γ. Let A, B, C, N be symmetric matrices of order n, and u a vector, and let $L(u)$ denote the operator

$$L(u)\equiv\frac{\partial}{\partial x}\left(A\frac{\partial u}{\partial x}+B\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(B\frac{\partial u}{\partial x}+C\frac{\partial u}{\partial y}\right)-Nu.$$

The elements of matrix N are assumed to be continuous functions of x, y, while the elements of A, B, C are assumed to possess continuous first derivatives. Matrices N and $\begin{Vmatrix} A & B \\ B & C \end{Vmatrix}$ are positive definite in D. The solution in D of the elliptic system $L(u)=f$ is said to be regular if $u\in C^2$ in D. Here f is a vector with components continuous functions of the variables. Let $\bar{u}$ be a vector defined on Γ with norm square summable. In the first paper cited above, the author shows that there exists at most one regular solution u of $L(u)=f$ in D such that

$$(*)\qquad \lim_{\alpha\to 0}\int_{\Gamma(\alpha)}|u-\bar{u}|^2ds=0.$$

A corresponding theorem for the parabolic system

$$L(u)-\partial u/\partial x=f$$

is also demonstrated. For the special elliptic system

$$\partial^2u/\partial x^2+\partial^2u/\partial y^2-Nu=f$$

it is proved that $|u|$ cannot have a relative maximum interior to D. As the title of the second paper implies many observations are made about solutions of the elliptic system $L(u)=0$. The main emphasis, however, being placed on proving the following theorem. If the relation (*) holds, then the solution u of $L(u)=0$ approaches along the normal to Γ the value there of $\bar{u}$ almost everywhere. Additional assumptions on the coefficients in $L(u)$ are used in proving this theorem, sufficient conditions being that the elements of the matrices belong to C^2. *F. G. Dressel.*

Green, J. W. An expansion method for parabolic partial differential equations. J. Research Nat. Bur. Standards 51, 127–132 (1953).

In the region S, $0\leqq x\leqq\pi$, $0\leqq t<\infty$, let u be a solution of $L(u)\equiv u_{xx}-u_t-gu=f$ meeting the initial and boundary conditions $u(x, 0)=u(0, t)=u(\pi, t)=0$. Here g and f are continuous in S together with their first two partial derivatives, and $f(0, 0)=f(\pi, 0)=0$. For each n, let $u_n(x, t)=\sum_{k=1}^{n}C_{n,k}(t)\sin kx$, where the $C_{n,k}(t)$ are determined by the conditions $\int_0^\pi\{L(u_n)-f\}\sin jx dx=0$, $C_{n,j}(0)=0$, $j=1, \cdots, n$. After showing u is unique in the class of functions belonging to C^2, the author proves $u_n\to u$, $u_{nx}\to u_x$, $u_{nt}\to u_t$ uniformly on bounded subsets of S. The functions u_{nxx} are shown to converge in the mean to u_{xx}, and an estimate of the error $|u_n(x, t)-u(x, t)|$ is given. *F. G. Dressel* (Durham, N. C.).

Widder, D. V. Positive temperatures on a semi-infinite rod. Trans. Amer. Math. Soc. 75, 510–525 (1953).

In Trans. Amer. Math. Soc. 55, 85–95 (1944) [these Rev. 5, 203] the author proved that any solution $u(x, t)$ of the heat equation, non-negative for $0<t<c$ and tending to zero as $t\downarrow 0$, is identically zero. Here the work is extended to the semi-infinite strip: a non-negative solution of the heat equation in $0<t<c$, $0<x<\infty$, which tends to zero as (x, t) approaches the coordinate axes, is identically zero. The example $xt^{-3/2}\exp(-x^2/4t)=u(x, t)$ shows that approach has to be taken in the two-dimensional sense. Let $k(x, t)=(4\pi t)^{-\frac{1}{2}}\exp(-x^2/4t)$, let $p>0$, let $\alpha(y)$ and $\beta(y)$ be monotone increasing functions such that $\int_{0+}^{1}yd\alpha(y)<\infty$ and $\beta(y)$ is positive. Then for $0<t<c$, $0<x<\infty$,

$$u_1(x, t)=pxt^{-1}k(x, t)+\int_{0+}^{\infty}[k(x-y, t)-k(x+y, t)]d\alpha(y),$$

$$u_2(x, t)=-\int_0^t k_x(x, t-y)d\beta(y),$$

are positive solutions of the heat equation, the former vanishing on the positive t-axis, the latter on the positive x-axis, and every positive solution with these initial properties is representable in such a manner. Finally, every solution which is positive in the semi-infinite strip is of the form $u_1(x, t)+u_2(x, t)$ where $u_1(x, t)$ and $u_2(x, t)$ have the representations given above. *E. Hille* (New Haven, Conn.).

Kasner, Edward, and De Cicco, John. The Fourier heat equation in Riemannian space. Proc. Nat. Acad. Sci. U. S. A. 38, 822–825 (1952).

The rate of flow of heat within a body in Riemannian space is given by a vector field $X^i(x, t)$, the temperature by a scalar function $U(x, t)$. Under the assumptions (1)

$X_i = -k\partial_i U$ (k is the conductivity) and (2) the rate of flow of heat into an n-dimensional region R is equal to the rate of increase of the temperature of R, it is found that U satisfies the Fourier equation $\operatorname{div}(k\partial_i U) = c\rho\partial U/\partial t$ (c specific heat, ρ the density function). There are in general ∞^2 heat surfaces $U(x, t) = \text{constant}$. The differential equation which has to be specified in order that $F(x, t, U) = 0$ represent this system is given in case $k/c\rho$ is constant. The system may consist of ∞^1 surfaces (degenerate heat family). $f(x) = c$ represents such a system if f obeys the equation $\operatorname{div}(\partial_i f) = af + b$ (a and b constants). The equations of Laplace, Poisson, and Helmholtz-Pockels are all of this form.

J. Haantjes (Leiden).

Fieber, H. Über die Temperaturverteilung in einem von stationärem Strom durchflossenen Draht. Österreich. Ing.-Arch. **7**, 161–168 (1953).

Let $T(r, z, t)$ denote the temperatures in a cylindrical wire $0 \leqq r \leqq R$, $-L/2 \leqq z \leqq L/2$, through which a uniform electric current flows. The ends of the wire are kept at the same constant and uniform temperature, the lateral surface undergoes heat transfer into surroundings at uniform temperature according to the usual linear law

$$k_1 T(R, z, t) + T_r(R, z, t) = K_1,$$

and the initial temperature is uniform. Assuming that the electrical conductivity varies linearly with the temperature and that the thermal coefficients are constant, the heat equation takes the form $T_t = a\nabla^2 T + bT + C$ where a, b, and C are constants. By applying the appropriate finite Hankel transformation in the variable r and the appropriate finite Fourier transformation with respect to z the formula for $T(r, z, t)$ is derived in series form. A concrete physical example is presented. *R. V. Churchill.*

Delavault, Huguette. Sur un problème de la théorie de la chaleur et sa solution au moyen des transformations de Fourier et de Laplace. C. R. Acad. Sci. Paris **237**, 1067–1068 (1953).

Let $f(t, x, y, z)$ denote the temperatures in a slab $0 \leqq z \leqq L$ of infinite extent when its initial temperature distribution and the temperature distribution on its faces $z = 0$ and $z = L$ are arbitrarily prescribed. The Laplace and double Fourier transformations and their convolution properties are employed here to derive a formula for f. *R. V. Churchill.*

Herbeck, M. Der Wärmeaustausch zwischen einem geheizten Band und einer Konvektionsströmung. Z. Angew. Math. Mech. **33**, 362–382 (1953). (English, French and Russian summaries)

A semi-infinite plate covering the half-plane $y = 0$, $x \geqq 0$, is kept at a unit temperature. A steady stream of air or other fluid passes over the plate in the region $y > 0$, the velocity of the stream having the direction of the x-axis and a magnitude proportional to y. The air is at temperature zero before it receives heat from the plate. Dimensionless coordinates x and y and temperatures $T(x, y)$ in the air stream are used. The problem of determining the steady temperature distribution $T(x, y)$ in the air stream is one of solving the differential equation $T_{xx} + T_{yy} = yT_x$ in the region $y > 0$ under the conditions $T \to 0$ as $y \to \infty$, $T(x, 0) = 0$ when $x > 0$, and some additional condition (C) on $T(x, y)$ at $y = 0$ when $x < 0$. A variety of conditions (C), for which at least approximate solutions are determined, are introduced. The flux of heat at the surface of the plate and the Nüsselt number are examined in each case and the results are compared with those of an experiment. The value of a free parameter in the boundary conditions is estimated in this way. The corresponding problem for a heated strip of finite width $-L/2 \leqq x \leqq L/2$ is treated concurrently with the above problem. *R. V. Churchill* (Ann Arbor, Mich.).

Difference Equations, Special Functional Equations

Efgrafov, M. A. A new proof of Perron's theorem. Izvestiya Akad. Nauk SSSR. Ser. Mat. **17**, 77–82 (1953). (Russian)

Perron's theorem is as follows: Suppose that the coefficients of the difference equation

$$(1) \qquad f(x+k) + a_1(x)f(x+k-1) + \cdots + a_k(x)f(x) = 0$$

satisfy (i) $\lim_{x\to\infty} a_m(x) = a_m$, $a_k(x) \neq 0$; (ii) $a_k \neq 0$; (iii) $\lambda^k + a_1\lambda^{k-1} + \cdots + a_k = \prod_{m=1}^{k}(\lambda - \lambda_m)$ and no two λ_m's have the same modulus. Then there exists a fundamental system of k solutions to (1), given by $f_1(x), \cdots, f_k(x)$, and

$$\lim_{x\to\infty} \frac{f_m(x+1)}{f_m(x)} = \lambda_m, \quad m = 1, 2, \cdots, k.$$

The author gives an extremely short, simple, and wholly elementary proof of this fact. *J. M. Danskin.*

Gel'fond, A. O., and Kubenskaya, I. M. On Perron's theorem in the theory of difference equations. Izvestiya Akad. Nauk SSSR. Ser. Mat. **17**, 83–86 (1953). (Russian)

[See the preceding review for notation.] Suppose that $\phi(x)$ satisfies (i) $\lim_{x\to\infty} \phi(x) = 0$; (ii) $0 < \phi(x+1) \leqq \phi(x)$; (iii) $\lim_{x\to\infty} \phi(x+1)/\phi(x) = 1$. The authors prove that if the hypotheses of Perron's theorem are satisfied and $|a_m(x) - a_m| \leqq \phi(x)$, $m = 1, 2, \cdots, k$, then there is a fundamental system of solutions of (1) satisfying

$$f_m(x+1)/f_m(x) = \lambda_m = O[\phi(x)], \quad m = 1, \cdots, k.$$

J. M. Danskin (Washington, D. C.).

Wouk, Arthur. Difference equations and J-matrices. Duke Math. J. **20**, 141–159 (1953).

The discrete analogue L_λ of a Sturm-Liouville operator is given by $L_\lambda x_n = \Delta(p_n \Delta x_n) + (q_n + \lambda)x_{n+1}$, where $x_0, x_1, \cdots$ is a sequence in the Hilbert space of sequences with $|x_0|^2 + \cdots < \infty$, $\Delta x_n = x_{n+1} - x_n$ and $p_0, p_1, \cdots$ and $q_0, q_1, \cdots$ are real sequences and $p_k > 0$. On the set of all $x = (x_0, x_1, \cdots)$ satisfying a real initial condition $x_0 \sin\phi + x_1 \cos\phi = 0$, L_0 is given by a Jacobi matrix $J = J_\phi$. If one J_ϕ is self-adjoint, so is the entire sheaf $(0 \leqq \phi < \pi)$. A list of partly known criteria for this case is given and it is shown that every real λ is in the spectrum of at least one J_ϕ and the motion of the isolated points of the spectrum is analytic in ϕ [this result was also obtained by Hartman and Wintner, Amer. J. Math. **72**, 775–786 (1950); these Rev. **12**, 717]. The limit process involved in the passage from the spectrum S_n of the nth section of a J-matrix to the spectrum S of the infinite matrix is treated with a view to the unsolved problem whether $\limsup S_n = S$ or $\limsup S_n \supset S$. For a periodic J, the first possibility holds. The paper ends with some applications to Mathieu functions. *L. Gårding* (Princeton, N. J.).

Aczél, János. Sur les équations fonctionnelles à plusieurs variables. II. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. 1 (1952), 311–333 (1953). (Hungarian. Russian and French summaries)

Ce travail donne la discussion de la solution de quelques équations fonctionnelles

$$z[z(x, y), z(u, v)] = z[z(x, u), z(y, v)], \tag{1}$$

$$z[z(x, y), u] = z[x, z(y, u)], \tag{2}$$

$$z[z(x, y), u] = z(x, y+u), \tag{3}$$

par réduction à des équations différentielles. Comme application de ce procédé, on obtient les différentes formes des conditions d'équations différentielles qui caractérisent les fonctions capables d'être représentées par un nomogramme à trois échelles linéaires. *Résumé de l'auteur.*

Kuwagaki, Akira. Sur la fonction analytique de deux variables complexes satisfaisant l'associativité :

$$f\{x, f(y, z)\} = f\{f(x, y), z\}.$$

Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **27**, 225–234 (1953).

With respect to the functional equation of the title (to be denoted by (1)) it is assumed that a complex number c exists (finite or infinite) such that $f(c, c) = c$ and also such that $f(x, y)$ has a Taylor expansion about (c, c). By a simple transformation c is carried to zero. The principal result is as follows: If $f(x, y)$ satisfies (1) and is holomorphic at $(0, 0)$ with $f(0, 0) = 0$, then f belongs to one of the four following categories: 1) $f \equiv x$; 2) $f \equiv y$; 3) $f \equiv x+y+xyS_1(x, y)$; 4) $f = xyS_2(x, y)$. Here S_1, S_2 are holomorphic at $(0, 0)$ and symmetric in x and y, and either $S_2(0, 0) \neq 0$ or $S_2(x, y) \equiv 0$. There are solutions of (1) in each of these classes. The proof makes extensive use of power series expansions.

I. M. Sheffer (State College, Pa.).

Hosszú, Miklós. On the functional equation of distributivity. Acta Math. Acad. Sci. Hungar. **4**, 159–167 (1953). (Russian summary)

Continuing the work of J. Aczél (not yet published) in characterizing strictly monotonic and twice differentiable solutions $F(x, y)$ of the functional equation

$$F[F(x, y), z] = F[F(x, z), F(y, z)],$$

the author determines the classes of strictly monotonic and twice differentiable solutions $F(x, y)$, $G(x, y)$ of

$$F[G(x, y), z] = G[F(x, z), F(y, z)].$$

E. F. Beckenbach (Los Angeles, Calif.).

Carlitz, L. A functional equation for the Weierstrass ζ-function. Math. Student **21**, 43–45 (1953).

It is shown that if $f(u)$, meromorphic in the finite plane, satisfies $\{f(u+v)-f(u)-f(v)\}^2 = -f'(u+v)-f'(u)-f'(v)$, then either $f(u) = -\frac{1}{6}b^2u+b$ (b = arbitrary constant) or $f(u) = \zeta(u) = \zeta(u; g_2, g_3)$ where ζ is the Weierstrass function.

I. M. Sheffer (State College, Pa.).

Doss, Shafik, and Nasr, Saad K. On the functional equation $dy/dx = f(x, y(x), y(x+h))$, $h>0$. Amer. J. Math. **75**, 713–716 (1953).

An existence theorem for the equation in the title. The f satisfies a Lipschitz condition and $|f(x, y, z)|$ for at least one (y, z) has a finite integral on (a, ∞). The proof is by successive approximations. *J. M. Thomas.*

Functional Analysis, Ergodic Theory

Nikodým, Otton Martin. On transfinite iterations of the weak linear closure of convex sets in linear spaces. Part A. Two notions of linear closure. IV. A study of convex sets in abstract linear spaces where no topology is supposed. Rend. Circ. Mat. Palermo (2) **2**, 85–105 (1953).

A detailed discussion of results given previously [C. R. Acad. Sci. Paris **234**, 1727–1728, 1831–1833 (1952); these Rev. **13**, 753]. *V. L. Klee, Jr.* (Seattle, Wash.).

Nikaidô, Hukukane. On a minimax theorem and its applications to functional analysis. J. Math. Soc. Japan **5**, 86–94 (1953).

Let E be a topological linear space, F a linear space, $X \subset E$ compact, $Y \subset F$, X, Y convex. Assume: (1) $K(x, y)$ is continuous on X for each $y \in Y$. (2) If $\alpha_1+\alpha_2 = 1$, $\alpha_1\alpha_2 \geqq 0$, $x_i \in X$, $y \in Y$ then $K(\alpha_1x_1+\alpha_2x_2, y) \geqq \alpha_1K(x_1, y)+\alpha_2K(x_2, y)$. If $\beta_1+\beta_2 = 1$, $\beta_1\beta_2 \geqq 0$, $y_i \in Y$, $x \in X$, then

$$K(x, \beta_1y_1+\beta_2y_2) \leqq \beta_1K(x, y_1)+\beta_2K(x, y_2).$$

Theorem 1. If $\sup_X \inf_Y K(x, y) < \infty$ then

$$\sup_X \inf_Y K(x, y) = \inf_Y \sup_X K(x, y).$$

The dual of this is also valid if we drop the topology in E, give F a topology and perform appropriate reflections on the statement (1) and Theorem 1. These results are used to derive (known) theorems about Banach spaces, namely Mazur's theorem, and the implication of reflexivity from uniform convexity or from the weak compactness of the unit sphere. Connections with the paper of H. Bohnenblust, S. Karlin and L. Shapley [Contributions to the theory of games, Princeton, 1950; pp. 181–192; these Rev. **12**, 514] are indicated. *B. R. Gelbaum* (Minneapolis, Minn.).

Lorentz, G. G., and Wertheim, D. G. Representation of linear functionals on Köthe spaces. Canadian J. Math. **5**, 568–575 (1953).

Les espaces de Köthe que considèrent les auteurs sont du type suivant : C étant un ensemble de fonctions mesurables $c(t) \geqq 0$ dans $[0, 1]$, normal, convexe, contenant la constante 1, contenant la limite d'une suite croissante de fonctions de C et tel que $\int_0^1 c(t)dt \leqq 1$ pour $c \in C$, l'espace de Köthe $X = X_C$ est défini comme l'ensemble des fonctions numériques mesurable $f(t)$ telles que $\|f\| = \sup_{c \in C} \int_0^1 |f(t)|c(t)dt < +\infty$; c'est un espace de Banach pour la norme $\|f\|$; cette catégorie d'espaces contient les L^p ($1 \leqq p \leqq +\infty$) comme cas particuliers. Pour un espace de Banach B, $X_C(B)$ est l'ensemble des applications mesurables f de $[0, 1]$ dans B telles que $|f(t)|$ appartienne à X_C. Désignant par B^* le dual de B, par X^* l'espace de Köthe conjugué de X (qui est aussi le dual de X moyennant une condition d'"absolue continuité" pour la norme $\|f\|$), les auteurs montrent que lorsque B est séparable et que X satisfait à la condition d'absolue continuité précitée, le dual de $X(B)$ s'identifie à $X^*(B^*)$. L'idée de la démonstration consiste à associer à une forme linéaire continue L sur $X(B)$ et à un point $x \in B$ la forme linéaire $f \to L(xf)$ sur X, qui est associée à un élément $g = U(x)$ de X^*, puis à déterminer la forme de $U(x)$ en faisant appel à un théorème de Kantorovitch caractérisant les opérateurs transformant la boule unité d'un espace de Banach en un ensemble latticiellement borné dans un L^p.

J. Dieudonné (Evanston, Ill.).

Zeller, Karl. Transformationen des Durchschnitts und der Vereinigung von Folgenräumen. Math. Nachr. **10**, 175–177 (1953).

Let A be the transformation $y=Ax$, $x=\{x_k\}$, $y=\{y_n\}$ given by the matrix $A=(a_{nk})$, so that $y_n=\sum_k a_{nk}x_k$. Let E_i, F_i be two decreasing sequences of "FK-spaces" of the author. Then A maps $\bigcap E_i$ into $\bigcap F_i$ if and only if for each j there is an i such that A maps E_i into F_j. There are three similar statements with one or both intersections replaced by unions. As an application, simple proofs of some results of Sheffer [Duke Math. J. **11**, 167–180 (1944); these Rev. **5**, 236] are obtained. *G. G. Lorentz* (Detroit, Mich.).

Bonsall, F. F. A note on subadditive functionals. J. London Math. Soc. **29**, 125–126 (1954).

Aronszajn [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **26**, 374–376 (1937)] showed that if p is a positive homogeneous, sub-additive functional on a linear space E such that (a) $p(x)+p(-x)>0$ for all $x\neq 0$, and if the space E is separable under the norm $\|x\|=p(x)+p(-x)$, then there is an additive homogeneous functional f on E such that (b) $f(x)<p(x)$ if $x\neq 0$. By an example in a non-separable real Hilbert space, the author shows that without separability (a) need not imply (b), even though the Hahn-Banach theorem does guarantee an f with $f(x)\leqq p(x)$ for all x in E. *M. M. Day* (Urbana, Ill.).

Orlicz, W. On the convergence of functionals representable as integrals over some classes of bounded functions. Studia Math. **13**, 208–217 (1953).

By appeal to his previous work in the theory of Saks spaces [Studia Math. **11**, 237–272 (1950); these Rev. **12**, 418] the author derives necessary and sufficient conditions for a sequence of functions $f_n(t)$ to be such that $\int_a^b f_n(t)x(t)dt$ converges for all $x(t)$ belonging to various subspaces of the space of all bounded functions, M. These subspaces are: K_1—functions continuous save at one point $t_0\in[a,b]$ (the same for all functions); K_2—functions continuous in $(-\infty,\infty)$; K_3—functions vanishing at $t_0\in[a,b]$ and of bounded variation in every pair of intervals $[a,t_0-\epsilon]$, $[t_0+\epsilon,b]$; K_4—functions of bounded variation on every finite interval. Consequences of the theory are as follows. 1. Let $K_n(s,t)$ be integrable in $[a,b]$ for every $s\in[a,b]$ and let $S_n(x,s)=\int_a^b K_n(s,t)x(t)dt$, where $x(t)\in M$, $x(t)$ measurable. Assume $x(t)\in M$ and $x(t)$ measurable implies $S_n(x,s)\xrightarrow{\text{ae}}x(s)$. Then there is a set $A\subset[a,b]$, $m(A)=b-a$, such that (a) for every $s_0\in A$, there is an $x(t)\in M$ which is continuous everywhere except possibly at s_0 and for which $S_n(x,s_0)$ diverges; (b) for every sequence $s_i\in A$, there is an $x(t)\in M$, continuous except possibly at all s_i and for which $S_n(s,s_i)$ diverges for all i. 2. Let $\varphi_i(t)$ be a CON system in $L_2[a,b]$, $S_n(x,s)=\sum_{i=1}^n(x,\varphi_i)\varphi_i(s)$. Then there is an $A\subset[a,b]$, $m(A)=b-a$ such that (a), (b) of 1 hold for A. Validity is maintained for any finite Toeplitz partial sums. *B. R. Gelbaum* (Minneapolis, Minn.).

Hille, Einar. Sur le problème abstrait de Cauchy. C. R. Acad. Sci. Paris **236**, 1466–1467 (1953).

Let U be a not necessarily bounded linear operator in a Banach space X. An abstract Cauchy's problem for U of order $n\geqq 1$ is defined by (1) $y^{(n)}(t)=U^n y(t)$ $(t>0)$ and $\lim_{t\to 0}\|y^{(k)}(t)-y_k\|=0$ $(k=0,\cdots,n-1)$. Here $y^{(k)}(t)$ denotes the kth strong derivative of $y(t)$. The solution $y(t)$ is said to be of normal type ω if

$$\limsup_{t\to\infty} t^{-1}\log\|y^{(n-1)}(t)\|=\omega<\infty.$$

The solution $y(t)\not\equiv 0$ is called a zero solution if it corresponds to the case $y_0=y_1=\cdots=y_{n-1}=0$. The author gives a necessary and sufficient condition for the existence of a zero solution of normal type ω: the eigenvalues of U^n (which is assumed to be a closed operator) are dense in $\{\lambda;\ |\lambda|(\cos(1/n\arg\lambda)^n>\omega\}$ and $U^n x(\lambda)=\lambda^n x(\lambda)$ has a solution $x(\lambda)$ which is bounded and regular for $\Re(\lambda)\geqq\omega+\epsilon$, $\epsilon>0$. When X is a Banach lattice with certain properties, the explosive solutions [E. Hille, Proc. Internat. Congress Math., Cambridge, Mass., 1950, v. 1, Amer. Math. Soc., Providence, R. I., 1952, p. 435] of (1) are discussed in connection with the solution of $Ux(\lambda)=\lambda x(\lambda)$: For a solution $x(\lambda)$ with a certain condition, $y(t)=\int_0^\infty \exp(\lambda(t-t_0))x(\lambda)d\lambda$ satisfies $y'(t)=Uy(t)$ for $t<t_0$ and $\lim_{t\uparrow t_0}\|y(t)\|=\infty$. *K. Yosida* (Osaka).

Michal, A. D. Completely integrable partial differential equations in normed linear spaces. Proc. Nat. Acad. Sci. U. S. A. **39**, 1089–1094 (1953).

Let N_1, N_2, N_3 be three complex Banach spaces, and let N_{23} be the Banach space of bounded linear transformations on N_2 to N_3. Let $f(x,y)$ be a function with values in N_3, defined for x and y in subsets of N_1 and N_2, respectively, and let $f(x,y;\delta x)$ denote the partial Fréchet differential of f with respect to x, if it exists. This differential is linear in δx, and hence defines an element of the space N_{13}. If this element is denoted by $f_x(x,y)$, so that $f_x(x,y)(\delta x)=f(x,y;\delta x)$, $f_x(x,y)$ may be called a partial derivative of f with respect to x. The partial derivative $f_y(x,y)$ is defined in similar fashion. It is a function of x and y with values in N_{23}.

Now consider a function $F(x,y,z,w,\delta x)$ with values in N_3, variables $x\in N_1$, $y\in N_2$, $z\in N_3$, $w\in N_{23}$, $\delta x\in N_1$, and a function $g(y)$ with values in N_3, variable $y\in N_2$. It is assumed that F is linear in δx and analytic in the quadruple (x,y,z,w) in a neighborhood of (x_0,y_0,z_0,w_0). Also, g is assumed to be analytic in a neighborhood of y_0, with $g(y_0)=z_0$ and $g_y(y_0)=w_0$. The problem posed in the paper is that of finding a function $z=f(x,y)$ such that

$$f(x,y;\delta x)=F(x,y,z,z_y,\delta x)$$

and $f(x_0,y)=g(y)$ for x and y sufficiently near x_0 and y_0 respectively. If N_3 is the space of complex numbers and N_1, N_2 are spaces of finite numbers of dimensions m, n respectively, with coördinate representations $x=(\xi_1,\cdots,\xi_m)$, $y=(\eta_1,\cdots,\eta_n)$, the foregoing Fréchet differential equation is equivalent to a system of m scalar equations:

$$\frac{\partial z}{\partial\xi_i}=F_i\left(\xi_1,\cdots,\xi_m,\eta_1,\cdots,\eta_n,z,\frac{\partial z}{\partial\eta_1},\cdots,\frac{\partial z}{\partial\eta_n}\right),\qquad i=1,\cdots,m.$$

If y and z_y are entirely absent, the form of the equation is that dealt with (in real Banach spaces) by Michal and Elconin many years ago [Acta Math. **68**, 71–107 (1937)]. In the present paper an existence and uniqueness theorem is stated and the proof is sketched. The conditions of complete integrability are too complicated to reproduce here. The method of proof is an adaptation of the classical Picard method.

A. Erdélyi prepared this paper for publication, using manuscript notes left by the late A. D. Michal. Erdélyi observes that there is no explicit proof that the solution $f(x,y)$ is analytic in x and y. However, once the existence of $f(x,y)$ with partial Fréchet differentials has been established, the analyticity of f jointly in x and y follows from what is known about the theory of abstract analytic func-

tions [see E. Hille, Functional analysis and semi-groups, Amer. Math. Soc. Colloq. Publ., v. 31, New York, 1948, Chapter IV and especially p. 90, Theorem 4.8.2; these Rev. 9, 594]. *A. E. Taylor* (Los Angeles, Calif.).

Halperin, I., and Nakano, H. Generalized l^p spaces and the Schur property. J. Math. Soc. Japan 5, 50–58 (1953).

Nakano [Proc. Japan. Acad. 27, 508–512 (1951); these Rev. 13, 954] defined, for A a set of indices, p and w functions on A such that $p(\alpha)\geq 1$ and $w(\alpha)>0$ for all α in A, a Banach space $l(p, w)$ consisting of all real-valued functions x on A which vanish at all but a countable subset of A and have finite $\|x\|=\inf t>0$ such that $\sum_\alpha w(\alpha)|x(\alpha)/t|^{p(\alpha)}\leq 1$. That paper gave conditions on the functions p_i necessary and sufficient that $l(p_1, w_1)$ and $l(p_2, w_2)$ be isomorphic; it was also shown that (a) for each $\epsilon>0$ the set of α with $P(\alpha)>1+\epsilon$ is finite is sufficient for the Schur property (b) each weakly convergent sequence in $l(p, w)$ is norm convergent. Those proofs used the theory of modulared semi-ordered spaces; the present note gives direct proofs, shows also that (a) is necessary for (b), and contains some extensions to spaces $l(p, w, B)$ where each $x(\alpha)$ has values in a Banach space B_α. *M. M. Day* (Urbana, Ill.).

Leader, Solomon. The theory of L^p-spaces for finitely additive set functions. Ann. of Math. (2) 58, 528–543 (1953).

Following Bochner [Ann. of Math. (2) 40, 769–799 (1939); these Rev. 1, 110] a family of spaces including the L^p spaces over spaces of measure one is defined and some of the familiar properties of L^p are proved. Let A be a Boolean algebra with unit e and let J be a positive, finitely additive function on J with $J(e)=1$. Let V^1 be the set of all real-valued, finitely additive functions on A which are absolutely continuous with respect to J. Let V^p, $1\leq p\leq\infty$, be the subset of V^1 consisting of those F for which $\|F\|_p=\sup_\Delta[\sum_{E\in\Delta}(F(E)/J(E))^pJ(E)]^{1/p}<\infty$, where Δ is a finite partition of e in A, if $p<\infty$; $\|F\|_\infty=\sup_E|F(E)|/J(E)$. Then, for $1\leq p\leq\infty$, V^p is a Banach lattice in which step functions are dense. If $1\leq p<\infty$ and if $1/p+1/q=1$, then $V^q=V^{p*}$. If $1<p<\infty$, then $\|F_\alpha-F\|\to 0$ in V^p if and only if $F_\alpha(E)\to F(E)$ for every $E\in A$ and $\|F_\alpha\|\to\|F\|$. If $2\leq p<\infty$, estimates are given for the rate of decrease of the coefficients in an orthogonal expansion in V^p. Some familiar properties of L^p spaces, such as the uniform convexity and the weak compactness of the unit sphere, are never mentioned in these general spaces. *M. M. Day* (Urbana, Ill.).

Grabiel, Federico. On a functional equation. Revista Soc. Cubana Ci. Fis. Mat. 3, 45–49 (1953). (Spanish)

The basic result is that if D is the set of all elements of $L_2(a, b)$ which have $n+1$ derivatives in $L_2(a, b)$, if T is a continuous linear functional on D, if dT/dx is defined by the equation $[dT/dx](f)=-T(f')$, and if T arises from a differentiable function g, then d^kT/dx^k arises from the kth derivative of g $(k=1, \cdots, n)$. *I. E. Segal.*

Sz.-Nagy, Béla. Sur les contractions de l'espace de Hilbert. Acta Sci. Math. Szeged 15, 87–92 (1953).

Let T be a contraction in the Hilbert space X. There is a Hilbert space Y containing X as a subspace and a unitary transformation U in Y such that $T^k=PU^k$, $(T^*)^k=PU^{-k}$ $(k=0, 1, 2, \cdots)$ where P is the orthogonal projection of Y onto X. The theorem has an analogue for one-parameter groups of contractions. From this theorem and the spectral theorem one can readily prove the following theorems concerning a contraction T on Hilbert space X: (i) (F. Riesz-B. Sz-Nagy) Every element invariant under T is also invariant under its adjoint. (ii) The mean ergodic theorem of von Neumann. (iii) (von Neumann-E. Heinz). If $u(z)$ is holomorphic for $|z|\leq 1$ then $u(T)$ is a contraction. *N. Dunford* (New Haven, Conn.).

***Inzinger, Rudolf. Drehungen vom Faltungstypus im Hilbertschen Raum und in der Menge der stützbaren Bereiche einer Ebene.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 354–360. Casa Editrice Perrella, Roma, 1953.

The author continues his studies [e.g., Monatsh. Math. 56, 105–125 (1952); these Rev. 14, 989] on the isomorphic systems H and L^2 of support functions in the plane H which are square-integrable on the interval $(0, 2\pi)$. Let $a(\phi)\in L^2$ and let $\{\alpha_\nu\}$ be the set of Fourier coefficients of $a(\phi)$, $\nu=0, \pm 1, \cdots$. Let $\{\tau_\nu\}$ be a sequence of complex numbers for which $\tau_{-\nu}=\bar{\tau}_\nu$. Then a linear transformation T which transforms $a(\phi)$ into a function $Ta(\phi)$ whose Fourier coefficients are $\{\tau_\nu\alpha_\nu\}$ is called a convolution transformation (ordinary convolution is of this type). If $\{\tau_\nu\}$ is bounded, so is T. The rotations of convolution type are defined by $|\tau_\nu|=1$, $\nu=0, \pm 1, \cdots$. The operation of differentiation D is of unbounded convolution type with $\tau_\nu=i\nu$. The author studies the operator $(E+sD)(E+s\bar{D})^{-1}$ and infinite products of such operators for various values of s. Connections with the theory of analytic functions are given. *E. R. Lorch* (Rome).

Birman, M. S. On the theory of self-adjoint extensions of positive definite operators. Doklady Akad. Nauk SSSR (N.S.) 91, 189–191 (1953). (Russian)

Some results completing those of Kreĭn [Mat. Sbornik 20(62), 431–495 (1947); these Rev. 9, 515] and Višik [Trudy Moskov. Mat. Obšč. 1, 187–246 (1952); these Rev. 14, 473] are announced. Let S be a closed positive operator in Hilbert space with a dense domain of definition $D(S)$. Let its lower bound $m(S)$ be positive and let S^* be its adjoint. According to Višik [loc. cit.] $D(S^*)=D(S)+S_\alpha^{-1}U+U$, where the sum is direct, U is the null-space of S^* and S_α is the "hard" self-adjoint extension of S [Friedrichs, Math. Ann. 109, 465–487 (1934)]. A part $\tilde{S}$ of S^* is self-adjoint if and only if $D(\tilde{S})=D(S)+(S_\alpha^{-1}+B)\tilde{U}_1+U_0$ where B is a self-adjoint operator defined on a subspace U_1 of U, $\tilde{U}_1=D(B)$ and $U_0=U\ominus U_1$. The author studies the extension $\tilde{S}$ in terms of B. Define B^{-1} by $D(B^{-1})=R(B)+U_0$, $B^{-1}U_0=0$. Then $(Sg, g)\geq\alpha(g, g)$ $(\alpha<m(S))$ for all g in $D(\tilde{S})$ if and only if $(Bv, v)\geq\alpha(v, v)+\alpha^2((S_\alpha-\alpha E)^{-1}v, v)$ for all v in $D(B^{-1})$. Let $D[\tilde{S}]$ be the closure of $D(\tilde{S})$ in the norm square $|g|^2=(\tilde{S}g, g)-\beta(g, g)$, $(\beta<m(S))$. Kreĭn [loc. cit.] proved that $D[\tilde{S}]=D[S_\alpha]+D[\tilde{S}]\cap U$. It is announced that the last term equals $D(B^{-1})$. If the negative spectrum of S is discrete, every eigenvalue has finite multiplicity and the origin is the only limit-point, then the negative spectrum of B has the same properties and conversely. Symmetric extensions of S are also studied in an analogous fashion. *L. Gårding* (Princeton, N. J.).

Kato, Tosio. On some approximate methods concerning the operators T^*T. Math. Ann. 126, 253–262 (1953).

In an earlier paper [J. Phys. Soc. Japan 4, 334–339 (1949); these Rev. 12, 447] the author gave estimates for the eigenvalue, and corresponding eigenvector, of a self-adjoint operator H in the interval (α, β) known to have a one-dimensional spectral manifold. These estimates were in

terms of a given trial vector $w \in \mathfrak{D}_H$, and were found by using $(\lambda-\alpha')(\beta'-\lambda)\geqq 0$ over $\alpha'\leqq\lambda\leqq\beta'$. Davis [Proc. Amer. Math. Soc. 3, 942–947 (1952); these Rev. 14, 659] has used n-fold products of such polynomials to get similar results. Temple [Proc. Roy. Soc. London. Ser. A. 211, 204–224 (1952); these Rev. 13, 691] has modified slightly Kato's procedure for an application to finite-dimensional dynamical systems. In the present paper the author proceeds in a different direction to relax the often irksome condition $w \in \mathfrak{D}_H$ in the special case $H=(T^*T)^{\frac{1}{2}}$ for T a closed linear operator with dense domain, so that H exists and is self-adjoint. Here the author gets estimates in terms of two trial vectors $u \in \mathfrak{D}_T$ and $u' \in \mathfrak{D}_{T^*}$. This is seen to be advantageous compared to the $\mathfrak{D}_H$ restriction in an example of a Sturm-Liouville eigenvalue problem. *F. H. Brownell.*

Glazman, I. M. On the theory of singular differential operators. Amer. Math. Soc. Translation no. **96**, 43 pp. (1953).

Translated from Uspehi Matem. Nauk (N.S.) **5**, no. 6(40), 102–135 (1950); these Rev. **13**, 254.

Faure, Robert. Sur certains opérateurs implicites donnant lieu à des intégrales premières. C. R. Acad. Sci. Paris **237**, 789–791 (1953).

When and only when the hermitian operators A and B commute are the eigenvectors of $(A-\lambda B)\varphi=0$ orthonormal. The eigenvectors φ_i and corresponding eigenvalues λ_i form then an "implicitly defined" mechanical quantity. The author gives the condition under which such a quantity is a constant of the motion. *L. Van Hove.*

Kesava Menon, P. On two positive forms of Arne Beurling. Math. Student **21**, 29–36 (1953).

The author proves without any use of Fourier analysis that if the form $\sum a_{\mu\nu}J_\mu\bar{J}_\nu$ is positive semi-definite, then so is $\sum \min(c_\mu, c_\nu)a_{\mu\nu}J_\mu\bar{J}_\nu$ for any positive c_μ, and strict positiveness can also be discussed. This type of theorem was first given by Beurling [Acta Math. **78**, 319–334 (1946); these Rev. **8**, 159] with the aid of Fourier transforms, and by employing such the author proves some further theorems as well. *S. Bochner* (Princeton, N. J.).

Van Hove, Léon. L'ensemble des fonctions analytiques sur un compact en tant qu'algèbre topologique. Bull. Soc. Math. Belgique **1952**, 8–17 (1953).

Let A_K denote the ring of complex-valued functions f, each analytic on some neighborhood of a compact subset K of the space of n complex variables $(z_1, \cdots, z_n)=z$. Reviewing a portion of an earlier paper [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 333–351 (1952); these Rev. **14**, 287], the author puts a topology t_K on A_K so that A_K becomes a linear topological space in which $f(z)$ and $\partial f(z)/\partial z_i$ $(i=1, \cdots, n)$ are continuous functions of f and z. Next, he shows that if K is a polyhedral domain [Cartan, Bull. Soc. Math. France **78**, 29–64 (1950); these Rev. **12**, 172], the automorphisms of A_K are the mappings of the form $(Af)(z)=f(\gamma z)$, where γ is a one-one analytic transformation of K upon itself. The derivations of A_K into itself are shown to be the usual linear differential operators of the first order. It then follows that all automorphisms and derivations of A_K are continuous in the topology t_K. *M. Henriksen* (Lafayette, Ind.).

Misonou, Yosinao. Operator algebras of type I. Kōdai Math. Sem. Rep. **1953**, 87–90 (1953).

§1. Quelques lemmes sur les projecteurs abéliens dans une W^*-algèbre. §2. Une W^*-algèbre A est de type I_α (α cardinal quelconque) si et seulement si A' est de multiplicité α. §3. Si A_1 est une W^*-algèbre de type I, *-isomorphe à A_2, et si A_1' est *-isomorphe à A_2', A_1 est unitairement équivalente à A_2. *J. Dixmier* (Paris).

Kaplansky, Irving. Modules over operator algebras. Amer. J. Math. **75**, 839–858 (1953).

Soient A une C^*-algèbre commutative à élément unité, H un A-module unitaire muni d'un produit scalaire (x, y) à valeurs dans A, linéaire en x, tel que

$$(x, x)\geqq 0, \quad (x, x)=0 \Rightarrow x=0, \quad (x, y)=(y, x)^*;$$

on norme H par $\|x\|=\|(x, x)\|^{1/2}$; H est "a kind of blend of a Hilbert space and a commutative C^*-algebra"; si H est complet, H est dit un C^*-module sur A. Les seuls "opérateurs bornés" considérés sur H sont ceux qui sont des endomorphismes de A-module. L'auteur concentre son attention sur le cas où A est une AW^*-algèbre, imposant alors quelques conditions supplémentaires ("algebraic substitute for a weak topology") et disant alors que H est un AW^*-module; il étudie les bases orthonormales, les sous-modules, les sommes directes de AW^*-modules sur A, les fonctionnelles continues sur H (à valeurs dans A), l'auto-dualité de H. L'algèbre B des opérateurs bornés sur H est (pour H fidèle) une AW^*-algèbre de type I de centre isomorphe à A; si H est "$\aleph$-homogène", B l'est aussi; ceci résout un problème d'existence posé par l'auteur [Ann. of Math. (2) **56**, 460–472 (1952); ces Rev. **14**, 291] (on notera que B n'est pas réalisé comme algèbre d'opérateurs dans un espace hilbertien), et l'auteur prouve aussi le théorème correspondant d'unicité: soient B une AW^*-algèbre de type I, e un projecteur abélien de B de support central 1; alors $H=eB$ est un AW^*-module sur $A=eBe$ (ceci fournit l'origine véritable des AW^*-modules), et la représentation régulière droite de B sur H fournit tous les opérateurs bornés de H. Applications: soit B une AW^*-algèbre de type I; 1) toute dérivation de B est intérieure; 2) tout automorphisme (non-nécessairement compatible avec l'adjonction) de B qui laisse fixe les éléments du centre est intérieur. *J. Dixmier* (Paris).

Helson, Henry. Isomorphisms of abelian group algebras. Ark. Mat. **2**, 475–487 (1953).

Let G and H be locally compact Abelian groups with character groups $\hat{G}$ and $\hat{H}$, respectively. If G and H are topologically isomorphic, then the group algebras $L_1(G)$ and $L_1(H)$ are clearly isometrically isomorphic. Partial answers to the converse problem are given in this paper. A) If T is a non-norm-increasing isomorphism of $L_1(G)$ onto $L_1(H)$, then G and H are topologically isomorphic under an isomorphism γ, and $Tf(y)=K\cdot\hat{\gamma}_0(y)f(\gamma^{-1}y)$ a.e. in H, for all $f \in L_1(G)$, where K is a constant depending on Haar measure in H and $\hat{\gamma}_0$ is a fixed character of H. B) If T is a continuous algebraic isomorphism of $L_1(G)$ onto $L_1(H)$, if $\|T\|<2$, and if $\hat{G}$ or $\hat{H}$ is connected, then the conclusion of A) obtains. Two preliminary theorems are of independent interest. C) A function ϕ defined on G is a Fourier-Stieltjes transform if and only if $\hat{f}\phi$ is an L_1 Fourier transform for all $\hat{f}$ which are L_1 Fourier transforms. D) If μ is a bounded Radon measure on G with purely discontinuous part $\sum_{k=1}^{\infty}a_k\epsilon_{p_k}$ (ϵ_p is the point measure $+1$ at p), and if the Fourier-Stieltjes transform $\hat{\mu}$ has absolute value 1 throughout $\hat{G}$, then $\sum_{k=1}^{\infty}|a_k|^2=1$. (This theorem for $G=(-\infty, \infty)$ is due to Beurling.) *E. Hewitt* (Seattle, Wash.).

Vulih, B. Z. Some questions of the theory of linear partially ordered sets. Izvestiya Akad. Nauk SSSR. Ser. Mat. **17**, 365–388 (1953). (Russian)

The present paper exposes completely and extends results announced earlier [Doklady Akad. Nauk SSSR (N.S.) **78**, 189–192 (1951); these Rev. **13**, 140]. Let X be a vector lattice (K-lineal is the author's term). X is said to be Archimedean if $x \in X$ and $x>0$ imply that $\{nx\}_{n=0}^{\infty}$ is an unbounded set. A non-void set $\{e_\xi\}_{\xi \in \Xi}$ of elements of X is called a fundamental system if $e_\xi>0$ for all $\xi \in \Xi$, $|e_{\xi_1}| \wedge |e_{\xi_2}| = 0$ for all distinct $\xi_1, \xi_2 \in \Xi$, and $\{e_\xi\}_{\xi \in \Xi}$ is maximal with respect to the preceding properties. Let Q be a compact Hausdorff space. The space $\mathfrak{C}_\infty(Q)$ consists of all continuous real-valued functions on Q which may assume the values $\pm\infty$ on nowhere dense sets. Scalar multiples in $\mathfrak{C}_\infty(Q)$ are defined in the natural way, and for $x, y \in \mathfrak{C}_\infty(Q)$, $x+y$ is defined as the function in $\mathfrak{C}_\infty(Q)$ which assumes the value $x(s)+y(s)$ at every point s where both x and y are finite. It has been proved earlier that $x+y$ exists and is unique [Kantorovič, Vulih, and Pinsker, Functional analysis in partially ordered spaces, Gostehizdat, Moscow-Leningrad, 1950, pp. 514–515; these Rev. **12**, 340]. The main theorem proved here is that every Archimedean vector lattice X is linearly and order-wise isomorphic to a linear subspace $\mathfrak{S}$ of $\mathfrak{C}_\infty(Q)$ for some compact Hausdorff space Q. Every element e_ξ in a fundamental system for X corresponds under this isomorphism to the characteristic function of an open and closed subset E_ξ of Q; the sets E_ξ are pairwise disjoint and their union is dense in Q. Every pair of distinct points of Q are functionally separated by a function in S. The space Q is completely determined up to homeomorphisms by the above specifications. If X is linearly and order-wise isomorphic to two subspaces $\mathfrak{S}$ and $\mathfrak{S}'$ of $\mathfrak{C}_\infty(Q)$, then there exists a homeomorphism of Q onto itself carrying $\mathfrak{S}$ onto $\mathfrak{S}'$. A number of other theorems, dealing largely with extensions of vector lattices, are also proved. A theorem similar to the one proved here has been published by K. Yosida [Proc. Imp. Acad. Tokyo **18**, 339–342 (1942); these Rev. **7**, 409]; it may be noted that Yosida does not prove Q compact or unique, but on the other hand treats the non-Archimedean case as well as the Archimedean.

E. Hewitt (Seattle, Wash.).

Vulih, B. Z. Generalized partially ordered rings. Mat. Sbornik N.S. **33(75)**, 343–358 (1953). (Russian)

Notation and terminology are as in the preceding review. Let X be an Archimedean vector lattice and let X contain an element x^* such that $x^* \wedge |x| > 0$ for all $x \neq 0$. Let 1, a unit of X, be a fixed element of this type. Let $\|x\| = \inf [\lambda; |x| \leq \lambda \cdot 1]$ if the indicated set is non-void; otherwise let $\|x\| = \infty$. An Archimedean vector lattice with unit complete with respect to convergence defined by this norm is called an ACE vector lattice. Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence of non-negative elements of X such that $x_m \wedge n \cdot 1 = x_n$ for $m \geq n$, $n = 1, 2, 3, \cdots$; if $\sup_{1 \leq n < \infty} x_n$ exists for every such sequence $\{x_n\}_{n=1}^{\infty}$, then X is said to be intrinsically normal. It is easy to see that X is intrinsically normal if and only if the representation space of functions $\mathfrak{S} \subset \mathfrak{C}_\infty(Q)$ for X (which can be chosen so that $1 \to$ the function identically 1) has the property that $x(s) \in \mathfrak{S}$, $0 \leq y(s) \leq x(s)$, and $y(s) \in \mathfrak{C}_\infty(Q)$ imply that $y(s) \in \mathfrak{S}$.

Previous communications of the author have shown how vector lattices satisfying stronger axioms than those of an ACE vector lattice may be made into algebras by giving an intrinsic definition of multiplication [Mat. Sbornik N.S. **22(64)**, 27–78, 267–317 (1948); these Rev. **10**, 46]. The results previously obtained are generalized partially to ACE vector lattices. To do this, the author defines a generalized partially ordered (GPO) ring. A set of elements Z is a GPO ring if: 1) Z is a lattice ordered [Abelian] group; 2) the product xy is defined for some pairs of elements $x, y \in Z$; 3) if xy and yz and at least one of $x(yz)$ and $(xy)z$ exist, then the other does also and $x(yz) = (xy)z$; 4) if xy and xz exist, then $x(y+z)$ exists and $x(y+z) = xy+xz$, and similarly for yx and zx; 5) there exists a unit i such that ix and xi exist for all x and $ix = xi = x$; if xy exists and $|x_1| \leq |x|$ and $|y_1| \leq |y|$, then x_1y_1 exists; 6) if $x, y \geq 0$ and xy exists, then yx exists. If xy exists implies that yx exists and $xy = yx$, then Z is said to be commutative.

The main theorem of the paper asserts that multiplication can be introduced into an ACE vector lattice X so that 1 becomes the unit and X becomes a GPO ring if and only if X is intrinsically normal. This can be done in exactly one way, and the ring so obtained is commutative. The property of intrinsic normality is obviously not invariant under a change of unit. Conditions under which a new unit yields an intrinsically normal vector lattice are obtained, and isomorphism of the GPO rings defined on X using these two units is established. Finally, the multiplicative linear finite real-valued functionals on an intrinsically normal ACE vector lattice are identified: they all have the form $x \to x(t_0)$, where X is represented as $\mathfrak{S} \subset \mathfrak{C}_\infty(Q)$ and $t_0 \in Q$.

E. Hewitt (Seattle, Wash.).

Calculus of Variations

Stampacchia, Guido. Sistemi di equazioni di tipo ellittico a derivate parziali del primo ordine e proprietà delle estremali degli integrali multipli. Ricerche Mat. **1**, 200–226 (1952).

Let $F(p_1, \cdots, p_n)$ be of class $C^{(2)}$ and convex. Suppose also that (i) $\mu \sum p_i^2 + m_1 \leq F(p) \leq M \sum p_i^2 + M_1$ and (ii) $|F_{p_i}(p)| \leq M \sum |p_i| + M_1$, where μ, m_1, M and M_1 are constants with $\mu>0$ and $M>0$. Let $u(x_1, \cdots, x_n)$ be a function of class $\mathfrak{A}^{(2)}$ [for this notation see the review of an earlier paper of Stampacchia, Ann. Mat. Pura Appl. (4) **33**, 211–238 (1952); these Rev. **14**, 483] on a region of class 2 in the sense defined by Miranda in the papers referred to below. Write $p_i(x) = \partial u(x)/\partial x_i$, and put

$$I(u) = \int_T F(p_1, \cdots, p_n) dx_1 dx_2 \cdots dx_n.$$

As is well known, $I(u)$ assumes a minimum among the set of all functions u of class $\mathfrak{A}^{(2)}$ which take on reasonably behaved preassigned values on FT. The author is concerned with the differentiability properties of the minimizing function u_0, properties which have been well worked out in the case $n=2$, notably by Hopf [Math. Z. **30**, 404–413 (1929)], Morrey [principally in Univ. California Publ. Math. (N.S.) **1**, 1–130 (1943); these Rev. **6**, 180; this paper contains an extensive bibliography] and Shiffman [Ann. of Math. (2) **48**, 274–284 (1947); these Rev. **9**, 45].

The two main results of the present paper are as follows. (I) u_0 has first partial derivatives which are AC with respect to each variable separately and second derivatives which are in L_2, and satisfies Euler's equation $\sum_{ij} F_{p_ip_j} \partial^2 u_0/\partial x_i \partial x_j = 0$. (Compare this with the remark of Shiffman, loc. cit., p. 283.) A related result holds when F involves x and u as well as p. (II) If $F(p)$ is analytic and u_0 has first partial

derivatives satisfying Hölder conditions with exponent between $\frac{1}{2}$ and 1, then u_0 is analytic. This generalizes a theorem of Hopf [loc. cit.].

The author leans heavily on the results of Miranda [Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I. (8) 3, 85–121 (1952); Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 85, 246–254 (1951); these Rev. 14, 175, 46].

J. M. Danskin (Washington, D. C.).

*Bertolini, Fernando. **Le variazioni successive ed il teorema di Jacobi, per un integrale dipendente da più funzioni di una sola variabile.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 8–22. Casa Editrice Perrella, Roma, 1953.

The author gives formulas for the variations of higher order of an integral $F(y)=\int_{x_1}^{x_2} f(x, y, y')dx$ in $(r+1)$-space (with fixed end-points), and derives necessary conditions, in terms of these higher variations, for a minimum of F in the case when the ends of the minimizing curve are conjugate. From this result are derived the known conditions on the envelope (in the case where a one-parameter family of extremals through the first end-point has an envelope). The corresponding case for the general problem of Bolza (i.e., the case when the second variation vanishes) has been treated by Morris Friedman [doctoral dissertation, Univ. of Chicago, 1952], who also derives sufficient conditions for a minimum. (His conditions involve no variations beyond the fourth.) *L. M. Graves* (Chicago, Ill.).

Mancill, Julian D. The Jacobi condition for unilateral variations in space. Univ. Nac. Tucumán. Revista A. 9, 15–22 (1952).

The author indicates how his treatment of the Jacobi condition for plane problems [Duke Math. J. 6, 341–344 (1940); these Rev. 2, 59] may be extended to problems in three-space. *L. M. Graves* (Chicago, Ill.).

Sakellariou, Neilos. On a problem of the calculus of variations. Prakt. Akad. Athēnōn 18 (1943), 115–122 (1950). (Greek)

The author studies one of the generalizations of the parametric problem of Bolza: to minimize among a class of arcs $x_i(s)$ $(s_1 \leqq s \leqq s_2)$ $(i=1, \cdots, n)$ satisfying (i) $x_i(s_1)=x_{i1}$ and (ii) $x_i(s_2)=x_i(\alpha_1, \cdots, \alpha_\rho)$, the parametric integral

$$J=\theta(\alpha)+\int_{s_1}^{s_2} F(x_i, x_{i_1}, x_i(s_2))ds.$$

Here F is positively homogeneous of degree 1 in the x_{i_1} and $\rho \leqq n$. He obtains some necessary conditions and calculates the second variation. *J. M. Danskin.*

Velte, Waldemar. Bemerkung zu einer Arbeit von H. Rund. Arch. Math. 4, 343–345 (1953).

It is shown that a transformation introduced by Rund [Arch. Math. 3, 207–215 (1952); these Rev. 14, 420] to derive a Hamiltonian function of the variational problem $\int f(x_i, \dot{x}_i)dt=\min.$ can be modified to obtain all Hamiltonian functions in the sense of Carathéodory. *S. Chern.*

Theory of Probability

Klamkin, M. S. On the uniqueness of the distribution function for the Buffon needle problem. Amer. Math. Monthly 60, 677–680 (1953).

Solomon, Herbert. Distribution of the measure of a random two-dimensional set. Ann. Math. Statistics 24, 650–656 (1953).

The random sets referred to are of the sort treated in the papers of Santaló [same Ann. 18, 37–49 (1947); these Rev. 8, 389] and Garwood [Biometrika 34, 1–17 (1947); these Rev. 8, 389], for example. In the general two-dimensional problem explicit conclusions have thus far been obtainable only for the first two moments. This paper gives more complete solutions for the special problem of one or two circular patterns dropped on a circular target.

L. J. Savage (Chicago, Ill.).

Takano, Kinsaku. A metrization of class-convergences of distributions. Ann. Inst. Statist. Math., Tokyo 5, 1–7 (1953).

Using Kunisawa's concentration function the author defines a (not easily describable) distance between two distribution functions (d.f.'s). Properties such as completeness of certain spaces of d.f.'s metrized by means of this definition are proved. The author is able, by means of this metric, to give a simple proof of a theorem of Khintchine [Izvestiya Naučno-Issled. Inst. Mat. Meh. Tomsk. Gos. Univ. 1, 261–266 (1937); also K. Takano, Ann. Inst. Statist. Math., Tokyo 3, 7–15 (1951); these Rev. 13, 566]. Generalizations of this theorem are proved. *J. Wolfowitz.*

Takano, Kinsaku. On the many-dimensional distribution functions. Ann. Inst. Statist. Math., Tokyo 5, 41–58 (1953).

The Lévy distance between two p-dimensional distribution functions (d.f.'s) $F(x_1, \cdots, x_p)$ and $G(x_1, \cdots, x_p)$ is defined as the smallest d such that

$$F(x_1-d, \cdots, x_p-d)-d \leqq G(x_1, \cdots, x_p) \leqq F(x_1+d, \cdots, x_p+d)+d$$

for all $x_1, \cdots, x_p$. The author shows that the principal properties of this distance which have been proved for one-dimensional d.f.'s apply to p-dimensional d.f.'s as well. Various results of the author's paper reviewed above for several metrics or topologies are extended to p-dimensional d.f.'s. *J. Wolfowitz* (Ithaca, N. Y.).

Dobrušin, R. L. Limit theorems for a Markov chain of two states. Izvestiya Akad. Nauk SSSR. Ser. Mat. 17, 291–330 (1953). (Russian)

Consider a Markov chain C_n with two states and the stationary transition matrix

$$P_n=\begin{pmatrix} p_n & q_n \\ \bar{q}_n & \bar{p}_n \end{pmatrix}$$

where $p_n+q_n=\bar{p}_n+\bar{q}_n=1$. Let ξ_n be the number of visits to state no. 1 in the first n steps in the chain C_n (double array scheme!). All possible conditions on the p's and q's where ξ_{n_k}, for an increasing sequence n_k and suitably normalized, has a non-degenerate limit distribution are determined and the various limit d. f.'s (of 7 types) are explicitly given. It turns out that only in one of the 7 cases does the limit depend on the sequence n_k, and then there are two limits depending on the parity of n_k. For instance, if $nq_n \to \infty$, $n\bar{q}_n \to \infty$ and $\max(np_n, n\bar{p}_n) \to \infty$ then the limit is normal; if $nq_n \to b < \infty$ and $p_n \to \bar{p} < 1$, then the limit is given in terms of Laguerre polynomials; etc. Other limit d. f.'s are expressible in terms of Bessel functions. In the proof local limit theorems are proved. The mass of results looks formidable but the methods seem to be elementary. Previous re-

sults by Kozulyaev [Učenye Zapiski Moskov. Gos. Univ. Matematika **15**, 181–182 (1939)] and B. O. Koopman [Proc. Nat. Acad. Sci. U. S. A. **36**, 202–207 (1950); these Rev. **11**, 445] are particular cases. [The reviewer does not understand (1.14): the second equality is obviously incorrect if $\rho(\{p_m\}, \{q_m\})$ is defined as it is a few lines above and below.] *K. L. Chung* (Syracuse, N. Y.).

Bellman, Richard. On a generalization of classical probability theory. I. Markoff chains. Proc. Nat. Acad. Sci. U. S. A. **39**, 1075–1077 (1953).

In probability one frequently considers a probability distribution (p.d.) as specified by a finite set of non-negative numbers of sum 1. The author replaces this p.d. by (b.p.d.) a finite set of real symmetric semidefinite matrices, $\{X_i, i \leqq N\}$, the sum of whose traces is 1. For the transformation by which a stochastic matrix replaces a p.d. by another is substituted the transformation

$$Y_i = \sum_{k=1}^{M} \sum_{j=1}^{N} A_{ijk} X_j A_{ijk}^*, \quad \sum_{i,k} A_{ijk} A_{ijk}^* = I,$$

which replaces a b.p.d. by another. The successive iterates of a stochastic matrix, as discussed in the theory of Markov chains then have an analogue in the successive applications of a transformation of the above type. It is stated that, if $\{X_i(n)\}$ is the result of n iterations acting on $\{X_i(0)\}$, and if A_{ijk} is positive definite for all i, j, k, and if certain degenerate cases are excepted, then $\lim_{n\to\infty} X_i(n) = X_i$ exists and is independent of $\{X_i(0)\}$. *J. L. Doob.*

Ottaviani, Giuseppe. Sulle catene doppie di Markoff. Giorn. Ist. Ital. Attuari **14**, (1951), 7–15 (1952).

Lévy [Colloques Internat. Centre Nat. Recherche Sci. no. 13, pp. 53–59 (1949); these Rev. **12**, 114] defined Markov processes with n-dimensional parameter ranges. The author exhibits such a process for $n=2$ which is non-degenerate in Lévy's sense. Lévy had given such an example but had shown that it was degenerate in a certain sense. The author does not investigate in what sense if any his example is degenerate. *J. L. Doob* (Urbana, Ill.).

Fuchs, Aimé. Sur quelques points de la théorie des processus de Markoff presque sûrement continus dans un intervalle. C. R. Acad. Sci. Paris **237**, 1137–1138 (1953).

The author states that the diffusion coefficients exist (as the usual limits) for any Markov process satisfying a certain continuity condition. (Presumably he has omitted further hypotheses, since his assertion, applied to a process defined by a single fixed continuous sample function, implies that every continuous function has a finite derivative almost everywhere.) Under further stated conditions, it is asserted that a Markov process is a solution of the stochastic differential equation studied by Ito [Mem. Amer. Math. Soc. no. 4 (1951); these Rev. **12**, 724]. *J. L. Doob.*

Fuchs, Aimé. Sur la continuité stochastique des processus stochastiques réels de Markoff. C. R. Acad. Sci. Paris **237**, 1388–1390 (1953).

Various definitions relating to sample functions of stochastic processes are given, and it is stated that they are equivalent for Markov processes. *J. L. Doob.*

Itô, Hirosi. Observed value of the autocorrelation function. Proc. Japan Acad. **29**, 198–202 (1953).

Let f be a stationary random process having a covariance $\varphi(\tau) = E[f(t+\tau)f(t)] = \int e^{i\tau\omega} dF(\omega)$. In an empirical determination of φ the delay time τ might be subject to fluctuations due to errors in measurement, so that the actual delay is $\tau+\xi$ where ξ is a random variable independent of f, τ and t. Let $\psi(\omega) = Ee^{i\xi\omega}$. It is shown that

$$E[f(t+\tau+\xi)F(t)] = \int e^{i\tau\omega}\psi(\omega)dF(\omega)$$

and that $E[f(t+\tau+\xi)f(t+\xi)] = \int e^{i\tau\omega}dF(\omega) = \varphi(\tau)$.
L. LeCam (Berkeley, Calif.).

Itô, Hirosi. Principle of the minimum entropy in information theory. Proc. Japan Acad. **29**, 194–197 (1953).

The theory of prediction in stochastic processes is usually based on a "least squares" type criterion. Using approximations the author attempts to replace this criterion by the corresponding "minimum entropy" criterion. The argument of the paper does not take into account the fact that, for prediction, only the past is known. *L. LeCam.*

Jánossy, L. Studies on the theory of cascades. Acta Phys. Acad. Sci. Hungar. **2**, 289–333 (1952). (Russian summary)

In the theory of nucleon cascades one has to consider processes in which a single particle of known energy, on passage through matter, gives rise to secondary particles of various energies, the number and variety of which as well as their energy distributions are random variables subject to laws of probability. A typical problem in the theory is to determine the probability of occurrence at a certain depth x of a definite number of particles of specified variety and with energies in specified intervals when a particle of known energy is incident at $x=0$. It is clear that it will be difficult to even formulate the relevant "diffusion equations" without some special notation. The present paper is principally devoted to describing a notation and developing a formalism which will enable the equations and the problems of the subject to be compactly formulated. The basic equation of the theory is one which the author calls the "G-equation." Simpler examples of the author's methods have been published earlier [Jánossy, Proc. Roy. Irish Acad. Sect. A. **53**, 181–188 (1950); these Rev. **13**, 569; Messel and Gardner, Physical Rev. (2) **84**, 1256 (1951); these Rev. **13**, 569].
S. Chandrasekhar (Williams Bay, Wis.).

Levert, Christoffel. La durée de retour et la durée d'absence de la plus grande valeur. C. R. Acad. Sci. Paris **237**, 374–376 (1953).

E. J. Gumbel [same C. R. **235**, 1598–1599 (1952)] has shown that the probability is approximately 2/3 that the greatest value corresponding to a return period T occurs between 0.32 T and 2.13 T. The author gives a proof of this statement which does not make use of extreme-value theory. *E. Lukacs* (Washington, D. C.).

Mathematical Statistics

Lewis, T. 99.9 and 0.1% points of the χ^2 distribution. Biometrika **40**, 421–426 (1953).

The lower 0.1% points of the χ^2 distribution have been computed correct to 8 decimal places for degrees of freedom $n=1(1)30(10)100,120$, and also the upper 0.1% points for $n=40(10)100,120$. The asymptotic formula for large n has been previously derived by Goldberg and Levine [Ann. Math. Statistics **17**, 216–225 (1946); these Rev. **8**, 42].
L. A. Aroian (Culver City, Calif.).

Kaarsemaker, L., and van Wijngaarden, A. Tables for use in rank correlation. Statistica, Rijswijk 7, 41–54 (1953). (Dutch summary)

Reviewed earlier as a report [cf. these Rev. 13, 664].

Mattila, Sakari. The decomposition of a series of observations by the method of iterated moving averages. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 156, 16 pp. (1953).

Given a series of nk equally spaced observations ($k=2m+1$)

$$[y_{\nu i}];\quad \nu=1, 2, \cdots, n,\quad i=1, 2, \cdots, k,$$

plus k observations both before and after this series. A procedure is suggested to test the following hypotheses. (1) The trend can be eliminated by a simple moving average; M: $k^{-1}(1, 1, \cdots, 1)$. (2) There is no periodic movement (of length k). The procedure is as follows. (i) Obtain the residuals from M: $(I-M)y$. (ii) Estimate the trend of these residuals: $(M-M^2)y$. (iii) Obtain the residuals from (ii): $(I-M)^2y$. (iv) The sum of squares of these residuals is divided into two parts: assuming M takes care of the trend, one is used to estimate the random variance, σ^2; the other, σ^2+ (periodic component). (v) The variation of the trend series (ii) is used to estimate σ^2+ (trend component).

The author intends to investigate the distribution functions of the three variance sources mentioned above in order to obtain the desired tests. *R. L. Anderson.*

Preston, Eric J. A graphical method for the analysis of statistical distributions into two normal components. Biometrika 40, 460–464 (1953).

Kamat, A. R. The third moment of Gini's mean difference. Biometrika 40, 451–452 (1953).

Sundrum, R. M. Moments of the rank correlation coefficient τ in the general case. Biometrika 40, 409–420 (1953).

The moments of orders up to four of the rank correlation coefficient τ [M. G. Kendall, Biometrika 30, 81–93 (1938)] are obtained for the case of a random sample from an arbitrary continuous bivariate distribution. A conjecture of H. E. Daniels and M. G. Kendall [ibid. 34, 197–208 (1947); these Rev. 9, 364] concerning the least upper bound of the variance of t in a sample from a finite population is confirmed. *W. Hoeffding* (Chapel Hill, N. C.).

Thomson, George Wm. Scale factors and degrees of freedom for small sample sizes for χ-approximation to the range. Biometrika 40, 449–450 (1953).

Quensel, Carl-Erik. The distribution of the partial correlation coefficient in samples from multivariate universes in a special case of non-normally distributed random variables. Skand. Aktuarietidskr. 36, 16–23 (1953).

It is known that the sampling distribution of the partial correlation coefficient $\rho_{ij\cdot 1,2,\cdots,r}$ from a normal multivariate population has the same distribution as the coefficient of correlation ρ except that n, the number of individuals in the sample, is replaced by $n-r$. The author proves this result under the following more general conditions. The set of random variables x_m ($m=1, 2, \cdots, p$) is divided into two sets x_k ($k=1, 2, \cdots, r$) and x_i ($i=r+1, r+2, \cdots, p$). The variables x_i are related to the variables x_k by the equation $x_i=\sum_{k=1}^{r}a_{ik}x_k+y_i$, where the random variables x_k are non-normally distributed and the random variables y_i are normally distributed and uncorrelated with the random variables x_k. In the general case $x_k=\sum a_{kl}x_l'$, where the x_l' are distributed in a Gram-Charlier Type A series.

L. A. Aroian (Culver City, Calif.).

Walker, A. M. The asymptotic distribution of serial correlation coefficients for autoregressive processes with dependent residuals. Proc. Cambridge Philos. Soc. 50, 60–64 (1954).

The main theorem is as follows. Let $X_1, X_2, \cdots, X_{n+u}$ be $n+u$ consecutive observations ($u\geq 0$) from the stationary autoregressive process $\sum_{i=0}^{s}\alpha_iX_{t-i}=Y_t$, where $\{Y_t\}$ is a stationary m-dependent process with $E(Y_t)=0$ and $E(Y_t^4)<\infty$. Then as $n\to\infty$, the distribution function of $Z_{n,u}=n^{-1/2}\sum_{t=1}^{n}(X_tX_{t+u}-E(X_tX_{t+u}))$ tends to that of a normal variate with zero mean and finite variance. This is an extension of a result of Diananda [same Proc. 49, 239–246 (1953); these Rev. 14, 771]. *W. Hoeffding.*

Birnbaum, Allan. Some procedures for comparing Poisson processes or populations. Biometrika 40, 447–449 (1953).

Let λ_1 and λ_2 be the parameters of two Poisson processes and define the random variable y_i so that $y_i=1$ (respectively $y_i=0$) if the ith event is observed in the first (respectively second) process. The y_i are a sequence of random variables having a Bernoulli distribution with

$$P(y_i=1)-p=\lambda_1/(\lambda_1+\lambda_2).$$

This is used to test the hypothesis $\lambda_1=\lambda_2$ against the alternative $\lambda_1\geq\lambda_2$. A confidence interval for the ratio λ_2/λ_1 is given and the duration of the experiment is also discussed.

E. Lukacs (Washington, D. C.).

Hamaker, H. C. "Average confidence" limits for binomial probabilities. Rev. Inst. Internat. Statistique 21, 17–27 (1953).

Let X be a chance variable with two possible values, 0 and 1, with probabilities $1-p$ and p respectively. Let k denote the sum of n independent observations on X. Given a number β ($0<\beta<1$), let $p_1(n, k)$ and $p_2(n, k)$ be two functions of k, with $p_1(n, k)<p_2(n, k)$ for all k, such that $P_p[p_1(n, k)<p<p_2(n, k)]\geq\beta$ for all p, exact equality holding for at least one p. The author notes that then $\int_0^1 P_p[p_1(n, k)<p<p_2(n, k)]dp>\beta$, and suggests replacing the confidence interval $[p_1(n, k), p_2(n, k)]$ by a shorter interval $[p_1'(n, k), p_2'(n, k)]$, where the functions $p_1'(n, k)$, $p_2'(n, k)$ are constructed so that

$$\int_0^1 P_p[p_1'(n, k)<p<p_2'(n, k)]dp$$

is approximately equal to β. Analogous suggestions are made for the case of an upper confidence limit alone and the case of a lower confidence limit alone. Tables of the suggested limits for various values of n and β are given.

L. Weiss (Charlottesville, Va.).

Downton, F. A note on ordered least-squares estimation. Biometrika 40, 457–458 (1953).

This note extends results of Lloyd [Biometrika 39, 88–95 (1952); these Rev. 14, 65] obtained for symmetric distributions to the unsymmetrical case. *B. Epstein.*

***Hansen, Morris H., Hurwitz, William N., and Madow, William G. Sample survey methods and theory. Vol. I. Methods and applications.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1953. xxii+638 pp. $8.00.

Chapter headings: 1. An elementary survey of sampling principles. 2. Biases and nonsampling errors in survey results. 3. Sample designs for some common sampling problems. 4. Simple random sampling. 5. Stratified simple random sampling. 6. Simple one- or two-stage cluster sampling. 7. Stratified single- or multi-stage cluster sampling. 8. Control of variation in size of cluster in estimating totals, averages, or ratios. 9. Multi-stage sampling with large primary sampling units. 10. Estimating variances. 11. Regression estimates, double sampling, sampling for time series, systematic sampling, and other sampling methods. 12. Case studies—designs and results of some actual sample surveys.

On the basis of large sample methods of point and interval estimation this book gives a comprehensive account of principles and methods of probability sampling for describing finite populations. The presentation is non-mathematical, references being made to Vol. II for derivations and proofs (cf. the following review). Biased estimates are favoured if they are consistent and have mean square errors that are smaller than those of corresponding unbiased estimates. Sequential methods are not dealt with, but the importance of exploiting available data or taking a preliminary sample when planning a survey is stressed. Much new material is presented, the main contributions being various extensions of the multistage sampling designs with varying probabilities developed by Hansen and Hurwitz [Ann. Math. Statistics **14**, 333–362 (1943); **20**, 426–432 (1949); these Rev. **5**, 210; **11**, 42]. Applications are made to the social field only. Somewhat loosely connected with the body of the book, a description of an acceptance sampling procedure for judging the quality of clerical work is included in the final chapter (without explicit references to the literature on acceptance sampling). The book contains about 70 exercises. *D. M. Sandelius* (Uppsala).

***Hansen, Morris H., Hurwitz, William N., and Madow, William G. Sample survey methods and theory. Vol. II. Theory.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1953. viii+332 pp. $7.00.

Chapter headings: 1. Introduction and definitions. 2. Fundamental theory—operations, events and probability. Fundamental theory—random variables, expected values, variances, covariances, and convergence in probability. 4–11. Same headings as in Vol. I (cf. the preceding review). 12. Response errors in surveys.

Except for Ch. 1 and 12 this book presents the mathematical background of Vol. I, but can be read as an independent book, Ch. 1 supplying the necessary statistical background. Ch. 2–3 give an excellent presentation of the probability basis of the theory of sampling from finite populations. Notable are, in particular, some theorems in Ch. 2 on conditional expectation, conditional variance and conditional covariance, which are used effectively in Ch. 4–12. Ch. 4–11 contain derivations and proofs of formulas and statements in Vol. I. Ch. 12 presents a theory (not available in Vol. I) for measuring and reducing a type of nonsampling errors. The book contains more than 80 exercises (excluding proofs left to the reader).

A large number of the proofs of Ch. 4–12 consist in determining the absolute minimum of a function of several variables subject to one (or, in one case, two) conditional equations. For the first of these minimization problems the solution obtained by Lagrangian multipliers is checked algebraically. This problem, the solution of which yields the Tschuprow-Neyman optimum allocation of a stratified sample, is, as was remarked by Tschuprow in 1923 in his paper referred to in the book, that of minimizing (*) $\sum_{i=1}^{t}(K_i/Z_i)$ subject to $\sum_{i=1}^{t}Z_i=1$, where the K_i are given positive constants. In the remaining cases no checks are made. Most of these cases are, however, of the type (*), while the others are more complicated.

D. M. Sandelius (Uppsala).

Sundrum, R. M. A method of systematic sampling based on order properties. Biometrika **40**, 452–456 (1953).

Rijkoort, P. J. Errata: A generalisation of Wilcoxon's test. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 407 (1953).

Cf. same Proc. Ser. A. **55**, 394–404 (1952); these Rev. **14**, 391.

Barnard, G. A. The frequency justification of sequential tests—addendum. Biometrika **40**, 468–469 (1953).

Cf. Biometrika **39**, 144–150 (1952); these Rev. **14**, 65.

Box, G. E. P. A note on regions for tests of kurtosis. Biometrika **40**, 465–468 (1953).

Cox, D. R. Some simple approximate tests for Poisson variates. Biometrika **40**, 354–360 (1953).

Let λ be the parameter of a Poisson process and let t be the time up to the nth count (n fixed). The quantity $2\lambda t$ is then approximately distributed as a chi-square with $(2n+1)$ degrees of freedom. This approximation is used by the author to derive tests for Poisson processes. A test is given for the hypothesis that two processes have the same parameters, also confidence intervals for the ratio of the parameters of two processes and for the parameter of a single process are constructed. *E. Lukacs* (Washington, D. C.).

Storch, J. M. Sequential sampling in the inspection of goods. Statistica, Rijswijk **7**, 3–14 (1953). (Dutch. English summary)

Expository paper.

Masuyama, Motosaburo. A rapid method of estimating basal area in timber survey—an application of integral geometry to areal sampling problems. Sankhyā **12**, 291–302 (1953).

Masuyama, Motosaburo. Rapid methods of estimating the sum of specified areas in a field of given size. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. **2**, no. 4, 113–119 (1953).

In the first paper the author reduces his forestry problem to the following geometrical form. It is required to estimate the total area ϕ, and also the number ν and the sum λ of perimeters, of a set of non-overlapping convex regions C contained in a plane region $\mathfrak{T}$ of area T. For any point P in $\mathfrak{T}$ let $k(P, r)$ be the number of regions C with dist $(P, C) \leqq r$, and $p(r)$ be the integral mean value of $k(P, r)$ over $\mathfrak{T}$. Using Steiner's formula [cf. W. Blaschke, Vorlesungen über Integralgeometrie, 2nd ed., Teubner, Leipzig-Berlin, 1936, formula (150*), p. 33] the author finds that

$$p(r) = (\phi + r\lambda + \nu\pi r^2)/T,$$

provided that $k(P, r)$ vanishes on the boundary of $\mathfrak{T}$, so that ϕ, λ, and ν can be determined if $p(r)$ is known for three different values of r. He suggests that $p(r)$ can be estimated by taking the average of $k(P, r)$ over a random sample of points P in $\mathfrak{T}$. He gives also some discussion of sampling errors involved, and (Appendix A) a simplified method that uses a sample of parallel straight lines (instead of points).

In the second paper he generalizes his method to cover other applications in which the regions C need not be convex. He applies three known formulae [cf. op. cit., formulae (194), (142), and (227), pp. 33, 24, and 37], which depend on integration with the kinematic density of a movable rigid arc (for example, a straight segment) as weight function, and he observes that the first and third formulae assume simpler and more easily applicable forms if the movements of the arc are confined to translations.

Both papers are illustrated by results of sampling experiments.

H. P. Mulholland (Birmingham).

Bartlett, M. S., and Rajalakshman, D. V. Goodness of fit tests for simultaneous autoregressive series. J. Roy. Statist. Soc. Ser. B. **15**, 107–124 (1953).

Bartlett and Diananda [same J. **12**, 108–115 (1950); these Rev. **12**, 512] have previously given a simplified method of derivation of Quenouille's goodness-of-fit test for an autoregressive stationary series with discrete time [J. Roy. Statist. Soc. (N.S.) **110**, 123–129 (1947); these Rev. **9**, 603]. This is here extended to the case of simultaneous autoregressive series. The tests are based on the asymptotic normality of certain linear forms in the sample auto- and cross-correlations. The tests are applied to two artificial series, yielding good agreement in one case but not in the other. The possible reasons for this lack of agreement are discussed and it is concluded that the original random numbers used for Series II [taken from L. H. C. Tippett, Random sampling numbers, Cambridge, 1927, p. 3] are in certain respects inadmissable. *D. G. Chapman.*

Kao, Richard C. W. Note on Miller's "Finite Markov processes in psychology". Psychometrika **18**, 241–243 (1953).

Goodman, Leo A. A further note on "Finite Markov processes in psychology". Psychometrika **18**, 245–248 (1953).

In a recent paper Miller [Psychometrika **17**, 149–167 (1952); these Rev. **14**, 188] discussed the problem of estimating the matrix T of transition-probabilities for a Markov chain when one is given the observed distributions among the a possible states, at times $t=1, 2, \cdots, n$, of a number of independent sample-sequences. If m_{ij} is the observed proportion in the ith state at the jth epoch, if M is the matrix of m's associated with the first $n-1$ epochs, if N is the similar matrix associated with epochs $2, 3, \cdots, n$, and if t_{ij} is the probability of a transition from state j to state i, Miller gave as the least-squares estimate, $\hat{T}=NM'(MM')^{-1}$ (if MM' is non-singular), but his proof was incomplete (this is the point of Kao's paper). Goodman supplies the missing argument and discusses some properties of $\hat{T}$. Miller had himself observed that his least-squares procedure may not be fully efficient, and Goodman remarks that it would be more appropriate to use the observed transitional proportions.

D. G. Kendall (Oxford).

Mathematical Economics

Følner, Erling. The elements of J. v. Neumann's theory of games. Nordisk Mat. Tidskr. **1**, 115–126, 144 (1953). (Norwegian. English summary)

Expository paper.

Danskin, J. M., and Gillman, L. A game over function space. Rivista Mat. Univ. Parma **4**, 83–94 (1953).

This paper is concerned with the solution of an example of a zero-sum two-person game with essentially bounded measurable functions on $(0, \infty)$ as pure strategies. The existence of a saddle-point is established by the fixed-point statement of Glicksberg [Proc. Amer. Math. Soc. **3**, 170–174 (1952); these Rev. **13**, 764] and the unique optimal strategies are characterized. *H. W. Kuhn.*

Bothwell, Frank E. The method of equivalent linearization. Econometrica **20**, 269–283 (1952).

The paper is an exposition of Kryloff and Bogoliuboff's method of "equivalent linearization" for the solution of highly nonlinear systems of differential (or difference) equations, illustrated by application to Goodwin's model of the business cycle [Econometrica **19**, 1–17 (1951)]. A periodic solution is assumed and substituted into the original equations; the coefficients and period are determined by equating coefficients, neglecting all higher harmonics. In effect, each nonlinear term is replaced by a quasi-linear one, where a periodic motion of the variables is transformed into a motion with the same period but whose amplitude is a non-linear function of the original amplitude. The stability of a given solution is then investigated by assuming a perturbation in the form of an exponential and then studying the real parts of the roots of the resulting characteristic equation. The procedure may be carried out graphically.

K. J. Arrow (Stanford, Calif.).

Dvoretzky, A., Kiefer, J., and Wolfowitz, J. On the optimal character of the (s, S) policy in inventory theory. Econometrica **21**, 586–596 (1953).

An (s, S) inventory policy is one which says to order for inventory when and only when current stock x has fallen below s, and then to order enough to bring the starting stock y up to $S \geqq s$. These (s, S) policies have received much practical use and theoretical attention [Arrow, Harris, and Marschak, Econometrica **19**, 250–272 (1951); these Rev. **13**, 368]. The present authors in an earlier paper [ibid. **20**, 187–222 (1952); these Rev. **13**, 856] showed that even in simple cases the optimal policy need not be (s, S). They now obtain necessary and sufficient conditions for the existence of an (s, S) policy which is optimal in a simple version of the inventory problem: expected loss is

$$V(x, y)=cy+A(1-F(y))+K_{yx},$$

where $c>0$ is carrying cost per unit of stock held; $F(z)$ is the known distribution function of demand, so $1-F(y)$ is the probability of not being able to meet all demands; $A>0$ is a penalty for default, independent of the extent of default; and $K_{yx}=0$ if $y=x$, $K_{yx}=K>0$ if $y>x$, a constant ordering cost independent of size of order. The authors characterize the classes of d.f.'s F for which an (s, S) policy is optimal a) for given c, A, K; b) for given c, A, and all K; c) for given A, K, and all c; d) for given c, K, and all A; e) for all choices of c, A, K. Not all results can be summarized here. For example, the class b) consists of all F's such that $V(x, x)$ is U-shaped or monotonically nondecreasing. The class e) consists of all F's concave for $y \geqq 0$. Details and geometry are given in the paper. *R. Solow* (Cambridge, Mass.).

TOPOLOGY

Morales Martinez, Rodolfo. On topologies for spaces of continuous functions. Bol. Soc. Mat. Mexicana **10**, 58–63 (1953). (Spanish)

This is a brief account of work done in recent years in the field mentioned in the title. Beyond this there are pointed out some relations not explicitly recognized in the literature; for example: a topology is proper if (besides the better-known only if) it is included in every admissible topology.

R. Arens (Princeton, N. J.).

Areškin, G. Ya. Free distributive lattices and free bicompact T_0-spaces. Mat. Sbornik N.S. **33(75)**, 133–156 (1953). (Russian)

The free distributive lattice $FD_0(\mu)$, where μ is a cardinal number ≥ 1, is a distributive lattice generated by a set M of elements having cardinal number μ, such that every single-valued mapping of M into a distributive lattice [reviewer's note: with μ generators] admits an extension over $FD_0(\mu)$ which is a lattice-homomorphism. Free lattices have been studied by Whitman [Ann. of Math. (2) **42**, 325–330 (1941), **43**, 104–115 (1942); these Rev. **2**, 244; **3**, 261]. The free distributive lattice with a finite number of generators has been identified by Th. Skolem [Tredje Skandinaviske Matematikerkongres, Kristiania, 1913, Aschehoug, Kristiania, 1915, pp. 149–163; see also G. Birkhoff, Lattice theory, Amer. Math. Soc. Colloq. Publ., vol. 25, rev. ed., New York, 1948, pp. 29–30, 145–146; these Rev. **10**, 673]. The author here constructs $FD_0(\mu)$ for an arbitrary cardinal number μ, as follows. Let M have cardinal number μ, let $\Sigma=\{\sigma\}$ be the family of all finite subsets of M, and let F be any finite subfamily of Σ such that $\sigma_1, \sigma_2 \in F$ imply $\sigma_1 \not\subset \sigma_2$. Then $FD_0(\mu)$ consists of all polynomials

$$a(F)=\bigvee_{\sigma\in F}\left[\bigwedge_{x\in\sigma} x\right],$$

where $a(F_1)\geq a(F_2)$ if and only if for every $\sigma_2 \in F_2$, there exists $\sigma_1 \in F_1$ such that $\sigma_2 \supset \sigma_1$. The lattice $FD_0(\mu)$ with adjoined O and I can be characterized in the following manner, which directly generalizes Skolem's theorem. Given $F \in \Sigma$, let $\bar{F}$ consist of all $\sigma \in \Sigma$ such that σ contains some $\sigma_1 \in F$. The lattice $\{FD_0(\mu), O, I\}$ consists of all such $\bar{F}$ under set-inclusion. As M. H. Stone has shown [Časopis Pěst. Mat. Fys. **67**, 1–25 (1937)], every distributive lattice L with O and I defines a certain compact T_0-space $T_0(L)$, whose points are the prime additive (or multiplicative) ideals of L, and a closed basis for which consists of all A_a ($a \in L$), where A_a is all prime additive ideals containing a. The remarkable result is obtained by the author that $T_0(FD_0(\mu))$ is homeomorphic with the Cartesian product of μ connected T_0-spaces each containing exactly two points.

E. Hewitt (Seattle, Wash.).

Toulmin, G. H. Uniform dimension and the product theorem. Quart. J. Math., Oxford Ser. (2) **4**, 198–203 (1953).

This paper introduces the concept of the uniform dimension of a topological space X [denoted by Dim X], and shows its relationship to the ordinary concept of the dimension of X [denoted by dim X]. Following the pattern of Hurewicz and Wallman [Dimension theory, Princeton, 1941, Definition III.1; these Rev. **3**, 312], the definition of the dimension of a subset over a given base is first introduced. If X is a topological space, $\{G\}$ a collection of its open sets, Y any subset (possibly equal to X), then Dim $Y/\{G\}=-1$ if and only if Y is the empty set; and Dim $Y/\{G\}\leq n$ if and only if $\{Y\cap G\}$ is a basis for Y and, for each G, Dim $F_Y(Y\cap G)/\{G\}\leq n-1$. Here $F_Y(Y\cap G)$ denotes the boundary of $Y\cap G$ in the subspace Y of X. Then Dim Y is defined as the least value of Dim $Y/\{G\}$ for any collection $\{G\}$ for which it exists. The integer Dim Y is independent of the containing space X, while Dim $Y/\{G\}$ depends only on Y and the collection $\{Y\cap G\}$.

Many of the usual properties of dimension hold for uniform dimension. If $Y\subset X$ and Dim $X/\{G\}=n$, then Dim $Y/\{G\}\leq n$. Moreover, Dim $X/\{G\}$ is not increased if $\{G\}$ is replaced by a subcollection which still forms a basis for X.

We always have dim $X\leq n$ if Dim $X=n$. In the case where X is separable and metric, the existence of either Dim X or dim X implies the existence of the other and the equality Dim $X=$dim X. The product theorem for uniform dimension (and hence for ordinary dimension in a separable metric space) can be proven without employing any kind of sum theorem. In connection with the theory of transfinite dimension [Hurewicz and Wallman, loc. cit., IV 6] the author has shown [Proc. London Math. Soc. (in the press)] that the closed-set sum theorem does not hold for transfinite dimension. He gives, in the present paper, a theorem which may be regarded as the natural generalization of the product theorem if the transfinite generalization of uniform dimension is employed.

D. W. Hall.

Plans, Antonio. An algebraic form of Urysohn dimension in Kuratowski space. Revista Acad. Ci. Zaragoza (2) **7** (1952), no. 2, 47–50 (1953). (Spanish)

Tits, J. Etude de certains espaces métriques. Bull. Soc. Math. Belgique **1952**, 44–52 (1953).

Following the attempt of Helmholtz, Lie and Kolmogoroff, the author gives a simple topological characterization of the various classical geometries. In fact, the author considers the pairs (R, G) with the following three properties: A) R is a connected, locally compact space, and G a transitive group of homeomorphisms of R; B) R has a uniform structure invariant under G; C) for $p, q \in R$, let us denote by G_p the isotropic subgroup of G at p and $O_p(q)$ the orbit of q under G_p. Then, for any two points q, r of R not in the same orbit under G_p, either $O_p(q)$ separates p and r or $O_p(r)$ separates p and q. All such pairs (R, G) are determined when R is finite-dimensional and G complete. With the exception of two cases O_1, O_2, they are given as follows: (i) R is the real euclidean (elliptic, hyperbolic) space, G is the group, or one of the open subgroups of the group, of all euclidean (elliptic, hyperbolic) motions; (ii) the counterparts of (i) for the complex field, the quaternion algebra and the Cayley algebra with the understanding that, in the last case, R is of 2 Cayley dimensions; (iii) R is a sphere S^n, G is either the group $\Gamma(n+1)$ of all orthogonal transformations or the group of all rotations. As for the exceptional case O_2, the author informs the reviewer that it should be changed as follows: R is the 8-dimensional real affine space, and G the group of affine transformations of the form $x\to x'+b$ where $x\to x'$ belongs to the faithful representation of degree 8 of the spinor group Spin (7) of order 7. Moreover, the pairs (S^8, G_2), $(S^8, I_2\times G_2)$, (P^8, G_2), $(S^7, \text{Spin (7)})$, $(P^7, \Gamma(7))$ should be added to the list where I_2 is the cyclic group of order 2, G_2 the compact simple Lie group of dimension 14,

and P^n the real projective space of dimension n. The invariant metrics under G are also discussed.

H. C. Wang (Auburn, Ala.).

Beckman, F. S., and Quarles, D. A., Jr. On isometries of Euclidean spaces. Proc. Amer. Math. Soc. **4**, 810–815 (1953).

The following theorem is proved: If T is a (possibly) one-many transformation of Euclidean n-space E_n, $n=2, 3, \cdots$, into itself, and if there is a positive number a such that for each pair of points p, q with $d(p, q)=a$ there exist $p'=Tp$ and $q'=Tq$ such that $d(p', q')=a$, then T is one-one and an isometry of E_n onto itself. *M. M. Day.*

Homma, Tatsuo. An extension of the Jordan curve theorem. Yokohama Math. J. **1**, 125–129 (1953).

A Y-set (or simple triod) is a set which is the union of three arcs pa, pb, pc, emanating from a single point p and intersecting pairwise only in p. An orientation of the 2-sphere S^2 induces, in a natural way, an orientation of each Y-set in S^2. The author proves the following generalization of the Schönflies theorem: Let M be a locally connected continuum in S^2 and let f be a homeomorphism of M into S^2. Then for f to be extensible to a homeomorphism of S^2 onto itself, it is necessary and sufficient that f preserve the orientation of every Y-set in M or reverse the orientation of every Y-set in M. *E. E. Moise* (Ann Arbor, Mich.).

Thomas, G. H. M. Simultaneous partitionings of two sets. Trans. Amer. Math. Soc. **75**, 69–79 (1953).

A partitioning of a set M is a finite collection G of disjoint connected open subsets $g_1, g_2, \cdots, g_n$ of M whose union is dense in M. If the diameter $\delta(g_i)$ of each g_i is $<\epsilon$, then G is an ϵ-partitioning of M. The mesh of G is $\max[\delta(g_i)]$. M is partitionable if for every $\epsilon>0$, M has an ϵ-partitioning. G is an S-partitioning if every g_i has property S (that is, if for every $\epsilon>0$, g_i is the union of a finite collection of connected sets of diameter $<\epsilon$). A sequence $G_1, G_2, \cdots$ of S-partitionings of M is decreasing if (1) G_{i+1} is a refinement of G_i for each i, and (2) the limit of the mesh of G_i is 0. If a partitioning G is a refinement of a partitioning H, and $h \in H$, then the core of h is the union of the closure of those elements of G whose closures lie in h; if for every $h \in H$, the core of h is connected and intersects the closure of every $g \in G$ that lies in h, then G is a core refinement of H. If $G=g_1, g_2, \cdots, g_n$ is a partitioning of M, and $N \subset M$, and $G'=\{g_1 \cap N, \cdots, g_n \cap N\}$ is a partitioning of N, then G is a simultaneous partition of M and N. The author's principal result is as follows: Let M be a compact metric partitionable set, and let N be a closed partitionable subset of M. Then there is a decreasing sequence $G_1, G_2, \cdots$ of simultaneous partitions of M and N, such that G_{i+1} (G'_{i+1}) is a core refinement of G_i (G_i') for each i. (This substantially extends a well-known theorem of Bing [Fund. Math. **36**, 303–318 (1949); these Rev. **12**, 348] to the effect that every compact metric locally connected continuum has a decreasing sequence of partitionings, in which each term after the first is a core refinement of its predecessor in the sequence.) The preliminary results include a large number of existence theorems for partitions with various special properties.

E. E. Moise (Ann Arbor, Mich.).

Strebe, David D. Irreducibly connected spaces. Duke Math. J. **20**, 551–561 (1953).

A study is made of the subsets about which a given connected set is irreducibly connected and of sets having base sets of certain types about which they are irreducibly connected. If a set M in a Hausdorff space is irreducibly connected about a subset B, but not about a proper subset of B, then B is called (Gehman) a basic set about which M is irreducibly connected. Sets having finite basic sets are studied and, among other results, a decomposition theorem for such sets is established. Likewise sets with countable basic sets are investigated and theorems proven from which it follows that any continuum having a countable basic set in this sense is necessarily an acyclic curve.

G. T. Whyburn (Charlottesville, Va.).

Lunc, A. L. The structure of the sets of points of local nonconnectivity and local connectivity of an arbitrary continuum. Mat. Sbornik N.S. **33(75)**, 463–470 (1953). (Russian)

The following theorem is proved. Let M be a metric space with a countable base. In order that there exist a continuum C such that M is homeomorphic with the set of points at which C is not locally connected, it is necessary and sufficient that M be an absolute F_σ and that $M=\bigcup_{n=1}^{\infty} F_n$, where each F_n is compact and has no trivial components. A proof of the following consequence is outlined. Let L be a metric space with a countable base. In order that there exist a continuum C such that L is homeomorphic with the set of points at which C is locally connected, it is necessary and sufficient that L be an absolute G_δ. *E. E. Floyd.*

Dyer, Eldon. Irreducibility of the sum of the elements of a continuous collection of continua. Duke Math. J. **20**, 589–592 (1953).

B. Knaster [Fund. Math. **25**, 568–577 (1935)] has shown that there is a continuous collection G of disjoint nondegenerate continua, such that G is an arc with respect to its elements as points, and such that the union G^* of the elements of G is a continuum which is irreducible between two of its points. The author shows that there does not exist a continuous collection G of disjoint decomposable continua whose union G^* is an irreducible continuum (so that, in particular, some element of Knaster's collection must be indecomposable). This substantially extends results of M. E. Hamstrom [Bull. Amer. Math. Soc. **58**, 204 (1952)] and the reviewer [ibid. **55**, 810–811 (1949); these Rev. **11**, 121]. *E. E. Moise* (Ann Arbor, Mich.).

Kapuano, Isaac. Sur les continus linéaires. C. R. Acad. Sci. Paris **237**, 683–685 (1953).

This note forms the completion of a previous note of the author [same C. R. **236**, 2468–2469 (1953); these Rev. **15**, 146] in which he made use of the concept of "chaînes emboitées". A one-dimensional continuum is said to be linear if and only if it is the limit of a collection of such chains. A closed chain is defined as a simple chain of open simply connected domains situated in R^n. A one-dimensional continuum is said to be linearly closed if and only if it is the limit of a collection of "chaînes fermées emboitées". If K is a linear continuum in R^n or a linearly closed plane continuum, then K is not homogeneous. Thus, to resolve a well-known problem of Knaster and Kuratowski there remains to be considered the case of the plane tree. The author also shows that every plane homogeneous linearly closed continuum is a Jordan curve. *D. W. Hall.*

Reifenberg, E. R. Fixed points on rotating continua. Proc. Cambridge Philos. Soc. **50**, 1–7 (1954).

Let I be a bounded plane continuum whose complement $\mathfrak{C}$ is a single domain and let T be a topological mapping of

the plane onto itself such that $TI=I$. Suppose that no prime end of $\mathfrak{C}$ is invariant under T. It is shown that if the frontier of I has a fixed point P, then all the fixed points in I are contained in any set X defined as follows. Let $\mathfrak{P}$ be a prime end containing P and let C be any curve in $\mathfrak{C}$ which converges to P and whose closure contains $\mathfrak{P}$. Then X is the union of the set $\bar{C}\cap I$ and its complementary interior domains. This theorem establishes a conjecture of Cartwright and Littlewood [Ann. of Math. (2) **54**, 1–37 (1951); these Rev. **13**, 148] who had obtained a slightly weaker result.

P. A. Smith (New York, N. Y.).

Kuratowski, K., et Steinhaus, H. Une application géométrique du théorème de Brouwer sur les points invariants. Bull. Acad. Polon. Sci. Cl. III. **1**, 83–86 (1953).

The n-dimensional case of the following theorem is proved. Let R be a bounded measurable set in the plane, and let $r=|R|$ be the measure of R. Let A_0, A_1, A_2 be the angles determined by three rays emanating from a point. Finally, let α_0, α_1, α_2 be three non-negative numbers such that $\alpha_0+\alpha_1+\alpha_2=1$. Then there is a translation τ such that, denoting τR by R', $|R'\cap A_i|=r\alpha_i$ for $i=0, 1, 2$.

E. G. Begle (New Haven, Conn.).

Reeb, Georges. Sur certaines propriétés topologiques des trajectoires des systèmes dynamiques. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8° **27**, no. 9, 64 pp. (1952).

A dynamical system (D.S.) is the pair formed by a manifold V_n and a vector field E in V_n (both of class C^∞). It is fibered (F.D.S.) if the trajectories of E are closed and define a fibering of V_n. A D.S. is called D.S.I., if it admits an integral invariant (Poincaré, Cartan), i.e. if $n=2q+1$, and there is given a differential 1-form π, with $\pi\wedge(d\pi)^q\neq 0$ at all points of V_n; E is determined by π. The author has studied these systems before [these Rev. **10**, 409; **12**, 196, 636; **13**, 653; **14**, 170, 584]; the present volume is an exposition and an extension of these results. Ch. 1 is a general introduction. Ch. 2 presents the formalism of differential algebra. Ch. 3 treats D.S.I., in particular the theorem on the non-existence of compact manifolds transversal to E. Ch. 4 treats F.D.S.I.; the base space of the fibering is a symplectic manifold V_{n-1}; relations between the Betti numbers of V_n and V_{n-1} are set up (equivalent to those derived from the Gysin sequence); a manifold with Poincaré polynomial $1+t^q+t^{q+1}+t^n$ cannot be an F.D.S.I.; etc. Ch. 5 treats Poincaré's method of small parameters; the vector field $E=E_\mu$ is assumed to depend on a parameter μ, such that for $\mu=0$ one has an F.D.S. A section is devoted to the case where the E_μ define D.S.I., and a connection with the Hamiltonian and with Morse theory is established. Ch. 6 is an exposition and extension of Seifert's theorem on closed trajectories of a vector field "near" to the vector field of an F.D.S. of dimension 3 [Proc. Amer. Math. Soc. **1**, 287–302 (1950); these Rev. **12**, 273]. Ch. 7 treats D.S. in the plane; characteristic exponents are investigated with the help of differential forms, which simplifies the computations; applications to Lienard's equation. Ch. 8 treats a special kind of D.S. (P.D.S.), where there are only a finite number of closed geodesics, with a non-degeneracy condition on characteristic exponents. Consideration of the stability character of the closed trajectories leads to a division of V_n into generalized "cells". Homology relations between these sets and V_n are established, yielding relations between the Betti numbers of V_n, and the number of closed trajectories with given stability characteristic, similar to Morse relations for critical points of a function; this is related to Thom's approach [C. R. Acad. Sci. Paris **228**, 973–975 (1949); these Rev. **10**, 558] to Morse theory (for a function on a manifold). As an application, a P.D.S. whose V_n is a homology-sphere, has at least $(n+1)/2$ closed trajectories.

H. Samelson (Princeton, N. J.).

Stone, A. H. On coverings of two-dimensional spaces. Proc. London Math. Soc. (3) **3**, 338–349 (1953).

Boltyanskiĭ defined the density $l(S)$ of a space S [Doklady Akad. Nauk SSSR (N.S.) **75**, 605–608 (1950); these Rev. **13**, 149, 1139] and showed that if S is a 2-dimensional compactum then $l(S)\geqq 6$. The author shows that, conversely, for a 2-dimensional normal space S one has $l(S)\leqq 7$. Furthermore, $l(S)=6$ if S is a sphere, torus, projective plane or Klein bottle, while $l(S)=7$ for any other closed surface. The structure of the nerve of a covering that "realizes" the density "minimally" is studied. For example, such a nerve is homeomorphic to S if S is neither the sphere nor the projective plane.

R. H. Fox (Princeton, N. J.).

Gugenheim, V. K. A. M. Piecewise linear isotopy and embedding of elements and spheres. I, II. Proc. London Math. Soc. (3) **3**, 29–53, 129–152 (1953).

Detailed exposition of results reviewed earlier [Proc. Nat. Acad. Sci. U. S. A. **38**, 333–337 (1952); these Rev. **14**, 74].

R. H. Fox (Princeton, N. J.).

Milnor, John. The characteristics of a vector field on the two-sphere. Ann. of Math. (2) **58**, 253–257 (1953).

Consider a smooth vector field in a region X of the 2-sphere bounded by a finite number of smooth curves without contact; assume that each critical point is a node, focus or saddle and that there are at most a finite number of closed characteristics. A closed characteristic polygon is either a closed characteristic or a cyclic sequence of compatibly oriented characteristics such that the end point of each is the beginning point of the following. A singular element is either a critical point or a boundary curve or a closed characteristic polygon together with a preferred side; it is called a source if the characteristics tend away from it and a sink if they tend towards it. A characteristic beginning or ending at a saddle point is called an asymptote. Then: (A) the number of sources at which no asymptotes begin equals the number of sinks at which no asymptotes end; (B) the number of asymptotes which begin at sources equals the number which end at sinks.

J. L. Massera.

Wallace, A. D. Cohomology, dimension and mobs. Summa Brasil. Math. **3**, 43–55 (1953).

By a mob, the author means a Hausdorff semigroup, i.e., a Hausdorff space with a continuous associative multiplication, and he studies the structure of such compact mobs. The main results are as follows: 1) Let S be a compact mob and P the set of those elements a in S satisfying $aS=S$. Furthermore, for any idempotent e of S, let $G(e)$ denote the unique maximal subgroup of S with e as its unit. Then, $ax=ay$ implies $x=y$ for any $a\in P$, $x, y\in S$, $ab\in P$ implies $a\in P$, $b\in P$, and P is the direct product of the submob of left units of S and the subgroup $G(e_0)$ of any left unit e_0. 2) Let S be a compact connected mob with left unit, T a closed ideal of S and Q a closed subset of T. Then the natural homomorphism $H^n(S, Q)\to H^n(T, Q)$ is an isomorphism onto for $n>0$ ($H^n=n$-dimensional cohomology group). It then follows that if, furthermore, S has a unit and if, for some $n>0$ and some coefficient group, $H^n(S)\neq 0$ but $H_p(A)=0$ for every closed proper subset A of S, then S

is a group. In particular, a compact manifold is a mob with unit only if it is a group. 3) Let S be a compact connected mob with unit u and let A be a closed subset of S such that $G(u)AG(u)\subset G(u)$ and that for some $n>0$ and some coefficient group there is an element h in $H^n(A)$ not extendable to $H^n(S)$ but extendable to $H^n(S_0\cup A)$ for any closed proper subset S_0 of S. Then $G(u)\subset A$. It then also follows that if an n-cell is a mob with unit u, $G(u)$ is contained in the boundary $(n-1)$-sphere.

Using a generalized notion of dimension, some more precise results are obtained. In particular, conditions are given for a compact subgroup of a compact connected mob S to coincide with the minimal closed ideal of S.

K. Iwasawa (Cambridge, Mass.).

Borsuk, K. Concerning the Cartesian division by manifold. Bull. Acad. Polon. Sci. Cl. III. 1, 91–94 (1953).

Continuing with his investigations into Cartesian products, the author shows that if C and C^* are two locally connected continua of dimension 1 and if M is a manifold, perhaps with boundary, then $C\times M$ and $C^*\times M$ are homeomorphic if and only if C and C^* are. Combined with an earlier result of the author, this yields the corollary that a factoring of a locally connected continuum Z in the form $Z=X_0\times X_1\times\cdots\times X_n$ is unique if $\dim X_0=1$ and if each X_i, $1\leq i\leq n$, is an arc or a simple closed curve.

E. G. Begle (New Haven, Conn.).

Moise, Edwin E. Affine structures in 3-manifolds. VII. Disks which are pierced by intervals. Ann. of Math. (2) 58, 403–408 (1953).

[For part VI see Ann. of Math. (2) 58, 107 (1953); these Rev. 14, 1003.] Let D in euclidean 3-space E be homeomorphic to a polyhedron. Then D is said to be tamely imbedded in E if E admits a self-homeomorphism mapping D onto a polyhedron. The definition extends at once to the case where E is replaced by a 3-manifold K, which will be taken to be triangulated. The author proves the following results: (1) If D is a topological disk pierced by an arc e at a point p, and if D and e are both tamely imbedded in K, then so is their union. (2) Let D be a disk pierced by a linear interval e on K at a point p. Let D be locally polyhedral on K at every point of $D-p$. Then $D\cup e$ is tamely imbedded. In fact, for each neighborhood U of p there is a homeomorphism f, of K onto itself, such that (i) f is the identity in e, and in $K-U$, (ii) f is piecewise linear over every finite polyhedron in $K-p$, and (iii) $f(D)$ is a polyhedron.

S. S. Cairns (Urbana, Ill.).

Weier, Josef. Fixpunkttheorie in topologischen Mannigfaltigkeiten. Math. Z. 59, 171–190 (1953).

Assuming the definition and properties of the index of an isolated fixed point of a mapping $E_n\to E_n$, the author defines the index of the fixed-point set of any mapping $f: M_n\to M_n$ where M_n is compact locally euclidean, and proves that the index is a homotopy invariant. Assume now that the fixed points of f are finite in number and that $n\geq 3$. If q and q' are fixed points belonging to the same Nielsen-Reidemeister class and U is an open set containing q and q' but no other fixed points, then f is homotopic to a mapping f' which coincides with f outside of U and has q for its only fixed point in U. For a certain class of triangulatable manifolds this had been proved by Wecken [Math. Ann. 118, 544–577 (1941); these Rev. 5, 275].

P. A. Smith.

Rodnyanskiĭ, A. M. Integral representations of the degree of a mapping. Doklady Akad. Nauk SSSR (N.S.) 91, 1019–1021 (1953). (Russian)

Denote by G a bounded region in Euclidean n-space R^n, and denote by G_φ the boundary of G. If $f: G\to R^n$ is continuous and each $f^{-1}(y)$ is compact, then Sitnikov [Mat. Sbornik N.S. 31(73), 439–458 (1952); these Rev. 14, 489] has defined the degree $\gamma(f, y)$ of f at the point y. In this paper, it is assumed that $f: \bar{G}\to R^n$ is continuous; for each $y\in R^n-fG_\varphi$, the degree $\gamma(f, y)$ is defined. Most of the theorems are listed below; no proofs are given. Theorem 1: $\gamma(f, y)$ is constant on each component of R^n-fG_φ (if O is such a component, $\gamma(f, O)$ will indicate this constant). Theorem 2: if f is differentiable on G and $J(x)$ is the Jacobian of f, then $\gamma(f, y)$ is characterized by the following: 1) $\gamma(f, y)$ is defined on R^n-fG_φ and constant on each component of R^n-fG_φ; 2) if $y\in R^n-fG_\varphi$ and $f^{-1}y$ contains no zero of $J(x)$, then $\gamma(f, y)=\sum_{x\in f^{-1}y}\operatorname{sign} J(x)$. Hereafter, suppose that f is continuously differentiable on G. Theorem 3: if O is a component of R^n-fG_φ with $J(x)$ summable on $f^{-1}(O)$, then $\gamma(f, O)=(\int_{f^{-1}(O)}J(x)dx)/\text{meas } O$. Hereafter, suppose that f is continuously differentiable on an open set containing $\bar{G}$ and that meas $G_\varphi=O$. Theorem 6:

$$\int_G J(x)dx=\sum\gamma(f, O_k)\cdot\text{meas } O_k,$$

where the summation is over all components of R^n-fG_φ. Theorem 7: if $\int_G J(x)dx\neq 0$, then fG_φ separates R^n. Theorem 9: if $\phi(y)$ is a real-valued summable function on fG, then $\int_G\phi(f(x))J(x)dx=\sum\gamma(f, O_k)\cdot\int_{O_k}\phi(y)dy$, where the summation is over all components of R^n-fG_φ.

E. E. Floyd.

Griffin, John S., Jr. Theorems on fibre spaces. Duke Math. J. 20, 621–628 (1953).

The author defines a fibre structure as a triple $\{X, B, \pi\}$, where π maps X onto B, such that an open covering, Ω, of B and a family of maps ϕ_U: $U\times\pi^{-1}U\to\pi^{-1}U$, $U\in\Omega$, exist verifying

$$\pi\phi_U(b, x)=b,\quad \phi_U(\pi x, x)=x,\quad b\in U,\quad x\in\pi^{-1}U.$$

The sets $\pi^{-1}b$ are called fibres; they need not be homeomorphic. However, a triple $\{X, B, \pi\}$ is called a bundle structure if an open covering, Ω, of B, a space Y and a family of homeomorphisms θ_U: $U\times Y\to\pi^{-1}U$ exist verifying $\pi\theta_U(b, y)=b$, $b\in U$, $y\in Y$. Hu has shown that every bundle structure is a fibre structure [Proc. Amer. Math. Soc. 1, 756–762 (1950); these Rev. 12, 435]. The author notes that every bundle structure is essentially an Ehresmann-Feldbau bundle, and, therefore, with suitably restricted fibre, a fibre bundle. In theorem 1 it is proved that a fibre structure is a bundle structure provided the fibres are homeomorphic and the pair (Ω, ϕ) may be chosen so that $\phi_U(b, \phi_U(a, x))=x$, $U\in\Omega$, $a, b\in U$, $x\in\pi^{-1}b$.

The rest of the paper is concerned with fibre structures with totally disconnected fibres. The author establishes the existence and uniqueness of covering paths and proves the first and second covering homotopy theorems without restriction on the domain of the map being deformed [cf. §11 of Steenrod, The topology of fibre bundles, Princeton, 1951; these Rev. 12, 522]. However, the lifting theorem for pairs $A, A_0\subset A$ (which would assert, for every pair of maps g: $A_0\to X$, h: $A\to B$, such that $h|A_0=\pi g$, the existence of a map $f: A\to X$, such that $f|A_0=g$, $\pi f=h$) is proved explicitly only when A_0 is a single point.

Further results give conditions on B under which the fibring admits a cross-section and under which the fibre

structure is a bundle structure. If the fibres are discrete and B is connected and locally path connected, then π is a covering map. *P. J. Hilton* (Cambridge, England).

Olum, Paul. Mappings of manifolds and the notion of degree. Ann. of Math. (2) **58**, 458–480 (1953).

In this paper the familiar notion of the degree of a mapping of one n-manifold into another is modified first by the concept of "twisted degree" and then by that of "group ring degree"; these concepts are used to study the homotopy classification problem for maps of a manifold M^n into a manifold Q^n for which $\pi_r(Q^n)=0$, $1<r<n$. We assume M^n and the universal covering manifold, $\bar{Q}^n$, of Q^n compact, since the problem is particularly simple in the contrary case.

For any manifold M^n with base-point x_0, the elements of $\pi_1(M^n, x_0)$ may be divided, in the obvious way, into the two classes "orientation-preserving" (o.p.), and "orientation-reversing" (o.r.) (the second class is, of course, void if M^n is orientable). A homomorphism $\theta: \pi_1(M^n, x_0)\to\pi_1(Q^n, y_0)$ is said to be orientation-true (o.t.) if it sends o.p. elements into o.p. elements, o.r. elements into o.r. elements. Let I be the additive group of integers; then $\pi_1(Q^n, y_0)$ acts on I by $\omega\alpha=\pm\alpha$, $\omega\in\pi_1(Q^n, y_0)$, $\alpha\in I$, where the sign is $+$ $(-)$ if ω is o.p. (o.r.). Let I^t be the resulting local group of "twisted integers" at y_0. Then the cohomology group with local coefficients, $H^n(Q^n, I^t)$, is cyclic infinite, generated, say, by v. Suppose a map f: $M^n, x_0\to Q^n, y_0$ induces θ, which is o.t.; then f also induces f^*: $H^n(Q^n, I^t)\to H^n(M^n, I^t)$; if the latter group is generated by u, then the twisted degree of f, deg f, is defined by $f^*v=(\deg f)u$. The twisted degree obviously coincides with the degree for maps of orientable manifolds.

The main theorem asserts that if $\pi_r(Q^n)=0$, $1<r<n$, then two maps $M^n, x_0\to Q^n, y_0$ inducing the same θ, which is o.t., are homotopic rel x_0 if and only if they have the same twisted degree. Moreover, for a given θ, the possible twisted degrees run over a single congruence class mod l, where l is the order of $\pi_1(Q^n)$ (if M^n or $\bar{Q}^n$ is not compact, the homotopy classes are in (1-1) correspondence with the elements of Hom $(\pi_1(M^n), \pi_1(Q^n))$).

If θ (induced by f) is not o.t., it is still possible to define a twisted degree, which is now an integer mod 2; however, it turns out in this case to be just the usual mod 2 degree. The author shows that, for each θ which is not o.t., there are exactly two classes of maps rel x_0 which induce it, and that these classes are sometimes distinguished by the twisted degree.

The concept of twisted degree is also applied to the study of the "free homotopy problem" (i.e., x_0 is allowed to wander during the homotopy). The results are more complicated and the methods of proof are based on an earlier paper by the author [Ann. of Math. (2) **57**, 561–574 (1953); these Rev. **14**, 895]. In theorem IIIa of that paper, representative (normal) maps were described for each homotopy class of maps of a locally-finite polyhedron P^n into a space Y for which $\pi_r(Y)=0$, $1<r<n$; these maps are studied in the present paper, in which P^n and Y are taken to be n-manifolds. Among other results, n-manifolds whose homotopy groups vanish in dimensions $1<r<n$ are classified by homotopy type; the generalized lens spaces of de Rham furnish an explicit example.

The concept of group-ring degree is introduced to increase the discriminating power of the twisted degree; thus it is equivalent to the twisted degree except if some o.r. element lies in the kernel of θ and the index $\theta(\pi_1(M^n, x_0))$ in $\pi_1(Q^n, y_0)$ is even. It is closely related to Hopf's Absolutgrad and is defined by using local coefficients from the local group I^r, which is the group ring of $\pi_1(Q^n, y_0)$ acted on, in the obvious way, by $\pi_1(Q^n, y_0)$. It solves the classification problem, where the twisted degree fails, for maps of projective n-space into itself (n even) which induce the trivial homomorphism of the fundamental group. *P. J. Hilton.*

Borel, A., et Serre, J.-P. Groupes de Lie et puissances réduites de Steenrod. Amer. J. Math. **75**, 409–448 (1953).

The authors study the reduced p-powers, due to Steenrod, for the cohomology of the classical groups and their classifying spaces; among the applications is the proof that the spheres S_{2n}, for $n\geqq 4$, do not admit almost complex structures [as announced in C. R. Acad. Sci. Paris **233**, 680–682 (1951); these Rev. **13**, 319]. Ch. I contains a brief exposition of the theory of universal bundles E_G and classifying spaces B_G of a compact Lie group G [cf. A. Borel, Ann. of Math. (2) **57**, 115–207 (1953); these Rev. **14**, 490]. The "universal" Chern, Pontryagin, and (highest) Stiefel-Whitney (Euler-Poincaré) classes (which determine these classes in all possible bundles) are constructed: The Chern classes are the generators of $H^*(B_{U(n)}, Z)$, identified with the elementary-symmetric functions in the (2-dimensional) generating variables of $H^*(B_{T(n)}, Z)$ (here $T(n)$ is the maximal torus of the unitary group $U(n)$); the Pontryagin classes mod p, p odd, are the generators of $H^*(B_{SO(n)}, Z_p)$, identified with the elementary-symmetric functions in the squares of the generators of $H^*(B_T, Z_p)$ (T the max. torus); for even n one has, instead of the highest Pontryagin class, the Stiefel-Whitney class, the product of the generators of $H^*(B_T, Z_p)$. Part of the construction is a proof of the duality theorem for Chern classes; it reduces to a simple identity between elementary symmetric functions. Ch. II presents a summary, without proofs, of the properties of the p-power operations of Steenrod. The formal rules allow to compute the p-powers for the complex and quaternion projective spaces. Applications to homotopy groups of spheres are made, by Steenrod's method: with $\alpha\in\pi_i(S_n)$ one attaches an $(i+1)$-cell to S_n and studies the p-powers in the space so obtained. Ch. III treats p-powers in Lie groups and classifying spaces. The basic idea is that, provided G and G/T (T = max. torus) have no p-torsion, $H^*(B_G)$ is a subalgebra of $H^*(B_T)$; B_T being a product of complex projective spaces, its p-powers are known from Ch. II; the problem for B_G reduces, for the classical groups, to calculation with symmetric functions. The p-powers for transgressive elements of $H^*(G)$ are determined by the fact that transgression and p-power commute. The p-powers of the Chern and Pontryagin classes [cf. Wu, Colloque de topologie de Strasbourg, 1951, no. IX, Bibliothèque Nat. et Univ. de Strasbourg, 1952; these Rev. **14**, 491] and of the Euler-Poincaré class are computed; the calculations parallel those of Wu. Ch. IV brings applications. For a large number of fiberings, with homology spheres as base spaces, the non-existence of cross-sections is proved, by the following principle: if f: $S_n\to X$, and if v, in $H^n(X)$, can be obtained, by cup-products and p-powers, from lower-dimensional elements of $H^*(X)$, then $f^*(v)$ is obviously 0; for f a (potential) cross-section of a bundle X over S_n, and v the image of the basic n-class of S_n under the projection this is a contradiction. This works, thanks to the computation in Ch. III, e.g., for $SU(n)/SU(n-1)$, $U(n)/U(n-1)$, $Sp(n-1)/Sp(n-2)$, for $n\geqq 3$, since there all elements of the relevant cohomology group of the bundle satisfy the condition above on v. Refined arguments produce non-zero

homotopy elements from the characteristic classes of these fiberings. The p-components of the homotopy groups π_i, for $i \leq 4p-3$, and π_i of the classical groups are computed. The characteristic class of $Sp(2)/Sp(1)$ is the Blakers-Massey element of $\pi_3(S_3)$; it is a generator of this group. Finally the complex Stiefel manifolds $W_{n,s}$ are treated; the p-powers follow from the fact that their cohomology is imbedded into that of the unitary groups. It is shown, e.g., that $W_{n,r+s}/W_{n-r,s} = W_{n,r}$ has no cross-section, unless either r or s is 1. *H. Samelson* (Princeton, N. J.).

GEOMETRY

*Dubnov, Ya. S. **Ošibki v geometričeskih dokazatel'stvah.** [Errors in geometrical proofs.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 68 pp. .95 rubles. No. 11 in the series "Popular lectures on mathematics."

*Lietzmann, W. **Der pythagoreische Lehrsatz. Mit einem Ausblick auf das Fermatsche Problem.** 7te Aufl. B. G. Teubner Verlagsgesellschaft, Stuttgart, 1953. 95 pp. DM 3.60.

Bilo, J. Sur les tétraèdres pseudoparallèles. Mathesis 62, 297–303 (1953).

Thébault, Victor. Tétraèdres supplémentaires. Mathesis 62, 289–293 (1953).

Goormaghtigh, R. Sur une généralisation du théorème de Droz-Farny. Mathesis 62, 293–296 (1953).

Chamberlin, Eliot, and Wolf, James. A decision method for trigonometric identities. Math. Mag. 27, 75–77 (1953).

Zacharias, Max. Über eine mit der Bydžovskýschen Konfiguration $(12_4, 16_3)$ verbundene Hessesche Konfiguration. Math. Nachr. 10, 187–196 (1953).

A $(12_4, 16_3)$ configuration consists of twelve points lying by threes on sixteen lines. The most symmetrical instance, which the author calls a Hessian configuration, is formed by sixteen lines a_i, b_i, c_i, d_i $(i=1, 2, 3, 4)$ with the rule that the four lines a_i, b_j, c_k, d_l are concurrent whenever $ijkl$ is an even permutation of 1234. Another instance was described by Bydžovský [Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. 1939, no. 2 (1940); these Rev. 7, 390; 13, 1138]. The author shows how a Hessian configuration can be derived from Bydžovský's. The twelve points consist of eight intersections of pairs of lines and four centers of perspective of pairs of triangles.

H. S. M. Coxeter (Toronto, Ont.).

Lauffer, Rudolf. Zur Topologie der Konfiguration von Desargues. II. Math. Nachr. 10, 179–180 (1953).

Continuing his discussion of the six types of general Desargues configuration [Math. Nachr. 9, 235–240 (1953); these Rev. 14, 1008], the author shows that the von Staudt polarity, which relates the ten points to the ten lines, is elliptic for type I, hyperbolic for all the others. The situation of the ten points with regard to the polarizing conic is as follows. For type II, four of the points are interior, and form a Hessian quadrangle (opposite sides conjugate). In two other cases, three are interior: a triangle for type III, a collinear triad for type V. For type IV, just two are interior, and for type VI, only one is. *H. S. M. Coxeter.*

*Pickert, Günter. **Analytische Geometrie.** Mathematik und ihre Anwendungen in Physik und Technik, Reihe A, Band 24. Akademische Verlagsgesellschaft, Geest & Portig K.-G, Leipzig, 1953. x+398 pp. DM 26.00.

The object of this book, as it appeared to the reviewer, is to give the reader a broad, self-contained and rather formal exposition of the fundamental concepts of analytical geometry rather than to penetrate to any depth into particular developments. This being so, it is natural that the book should be mainly about n-dimensional geometry, the excursions into plane or 3-dimensional geometry assuming to a large extent the role of illustrative applications. The book is in three main sections, respectively entitled Affine geometry, Metrical geometry, and Projective geometry. Section I (pp. 1–129) develops axiomatically the properties of affine spaces and of vector spaces generally, first over the field of real numbers and then over any skew field; and it then proceeds to a long exposition of algebraic fundamentals: properties of linear algebra, linear transformations, determinants and matrices, eigenvalues and eigenvectors, etc. Section II (pp. 130–226) gives an account of metric spaces, based on the use of inner products; and it deals at length with the general concept of angle, motions, content, with further sub-sections on the circle and sphere, complex metric spaces, and the classification of real quadrics. Section III (pp. 227–361) outlines the fundamentals of projective geometry: cross ratios, collineations and correlations, projections (for any skew ground field), quadrics, symmetrical and skew-symmetrical polarities; and after two further sub-sections on the derivation of affine and metrical geometries from projective geometry, the book ends with a brief account of line-geometry in S_3 and the representation of lines of S_3 by points of a quadric in S_5. There are brief appendices dealing with the arithmetic definitions of trigonometric functions, the pure axiomatic structure of plane projective geometry, and automorphisms of the field of real numbers. A useful index is provided. Throughout the book there are copious sets of examples, for use not only as exercises but also as brief indications of important developments that could not be included in the main text.

The reviewer found the general style of the exposition a little heavy and the rather solid set-up of the text not very attractive. The argument itself, on the other hand, seemed to be uniformly meticulous and clear. The book should be of great value to a reasonably resolute beginner who wants a systematic and self-contained exposition of the broad foundations of modern analytical geometry.

J. G. Semple (London).

Fulton, Curtis M. Catenary and tractrix in non-Euclidean geometry. Math. Mag. 27, 79–84 (1953).

Adams, George. Conchoid and negative circle. Elemente der Math. 8, 129–136 (1953); 9, 9–16 (1954).

*Sen, R. N. **A course of geometry.** Calcutta University Press, Calcutta, 1953. xxxvi+311 pp. 12 rupees.

This is a simple but rigorous treatment of the following topics: vector spaces, matrices, systems of linear equations, determinants, linear transformations, group theory (a brief introduction), vectors and angles in the plane, the area of a triangle, cross-ratio, harmonic conjugates, rigid motions, conics (general and special equations, focus and directrix), Laguerre's definition of angle, homothetic and congruent

transformations (with the unusual name "paddle motion" for a glide-reflection), inversion, polar reciprocation, affinities (including simple strains and shears), classification of conics, conjugate diameters, projective pencils of lines, involutions, the quadrangle and quadrilateral, homogeneous coordinates, collineations and correlations, linear and quadratic dependence, Pascal and Brianchon, pencils of conics, invariants of a pair of conics, trilinear and areal coordinates, vectors in three dimensions, lines and planes, duality, line-coordinates, transversals of three skew lines, homology, Desargues' theorem, groups of transformations, the Erlangen program, reguli and cones, polarities and null-systems, the circle at infinity, orthogonal transformations, classification of quadrics, confocals, and the reduction of the general quadratic form to its normal form.

As the above outline shows, more ground is covered here than in most elementary texts. The book is well printed and has good diagrams, instructive examples, and an index.

H. S. M. Coxeter (Toronto, Ont.).

***Goodstein, R. L., and Primrose, E. J. F. Axiomatic projective geometry.** University College, Leicester, 1953. xi+140 pp. 15s.

Der Titel des Buches lautet nicht "Axiomatik der projektiven Geometrie" sondern "Axiomatische projektive Geometrie"; es liegt demgemäss nicht in der Absicht des Verf., ein Axiomensystem aufzustellen, es auf Widerspruchsfreiheit und Unabhängigkeit zu untersuchen und seine Vollständigkeit zur Herleitung der Sätze der projektiven Geometrie darzutun. Es werden vielmehr dogmatisch 10 Axiome an die Spitze gestellt und daraus die ebene projektive Geometrie der Punkte, Geraden und Kegelschnitte synthetisch hergeleitet. Diese Axiomen enthalten weder Anordnungs- noch Stetigkeitsaussagen. Sie umfassen insbesondere die ebenen projektiven Verknüpfungssätze und die Nichtkollinearität der drei Nebenecken des vollständigen Vierecks. Sie enthalten die Forderung, dass eine projektive, d.h. durch eine Kette von Perspektivitäten definierte Verwandtschaft zwischen Grundgebilden 1. Stufe durch 3 entsprechende Paare eindeutig bestimmt sei. Punkte, Geraden und Kegelschnitte erscheinen als selbständige Grundgebilde, auch wenn letztere axiomatisch als Erzeugnis projektiver Büschel definiert werden.—Die Betrachtung komplexer Punkte wird erstmalig notwendig bei der Frage nach Fixelementen einer projektiven Abbildung eines Gebildes auf sich bezw. nach Schnittpunkten zweier Kegelschnitte. Die Existenz eines Fixpunktes wird axiomatisch gefordert; ebenso die Existenz eines gemeinsamen Punktes von zwei Kegelschnitten. Haben diese in dem gemeinsamen Punkt verschiedene Tangenten, wird ein weiterer Schnittpunkt der beiden Kegelschnitte axiomatisch gefordert.

Das letzte Kapitel des Buches skizziert eine Formalisierung der Aussagen der ebenen projektiven Geometrie und ihrer Beweise, wobei auf die formale Logik nicht Bezug genommen wird. Jede Aussage wird dargestellt durch ein endliches Schema aus 7 Spalten, die mit Symbolen für Punkte besetzt sind. Jede Spalte charakterisiert eine geometrische Aussage, wie: Punkte fallen zusammen; fallen nicht zusammen; liegen perspektive; liegen auf einem Kegelschnitt u.ä. Die geometrischen Axiome ergeben Verwandlungsregeln für die Elemente eines solchen Schemas. Am Beweis der Sätze von Pappus und Pascal wird dieser Formalismus erläutert.

Das Buch soll den Anfänger mit dem axiomatischen Aufbau einer mathematischen Disciplin vertraut machen, wobei der Zahlbegriff bewusst vermieden wird. Das Buch enthält 128 Übungsaufgaben zur Vertiefung des Stoffes, jedoch keine Literaturhinweise. Inhalt der 10 Kapitel: Homography; The harmonic construction; The conic; Amicable involutions; The fundamental involution; Complex geometry; The circular points; The common points of two conics; The reduction of a chain of perspectivities; Geometry as a board game. *R. Moufang.*

Wyler, Oswald. Incidence geometry. Duke Math. J. **20**, 601–610 (1953).

An incidence geometry is essentially a geometry in which the lines and pencils of lines on a point are the points and lines of a projective geometry. This article is a continuation of the study of incidence geometries begun by Gorn [Bull. Amer. Math. Soc. **46**, 158–167 (1940); these Rev. **1**, 165]. The principal contributions are: (1) a new definition (given both synthetically and lattice-theoretically), not requiring finite-dimensionality; (2) a reducible incidence geometry (some pencil of lines on a point contains exactly two lines) is a reducible projective geometry; (3) a new proof of Gorn's theorem that an incidence geometry of more than three dimensions is embeddable in a projective geometry. The author concludes with an unsolved problem relating to the embedding of three-dimensional incidence geometries.

A. J. Hoffman (Washington, D. C.).

Forder, H. G. On gauge constructions and a letter of Hjelmslev. Math. Gaz. **37**, 203–205 (1953).

Hjelmslev communicated to the author a proof for the following theorem of absolute geometry on the basis of congruence axioms only: Let OAB, OBC, $O'A'B'$, $O'B'C'$ be right angles and $OA=O'A'$, $AB=B'C'$, $BC=A'B'$; then $OC=O'C'$. The author had noticed that this theorem was used implicitly by Hjelmslev [Mat. Tidsskr. B. **1943**, 21–26; these Rev. **7**, 69]. *H. Busemann* (Los Angeles, Calif.).

***Menger, Karl. Géométrie générale.** Mémor. Sci. Math., no. 124. Gauthier-Villars, Paris, 1954. 80+i pp. 1000 francs.

The brief review of selected topics of Distance Geometry contained in this Mémorial formed the basis of lectures given by the author at the Sorbonne in 1951. It is similar in style and spirit to earlier short summaries of distance-theoretic results published by its author [Jber. Deutsch. Math. Verein. **40**, 201–219 (1931); Enseignement Math. **35**, 348–372 (1936)]. The first three chapters (Geometry in metric spaces, General theory of curvature, Analysis and generalization of the notion of metric space) deal with aspects of the development of Distance Geometry itself, while the results discussed in the remaining four chapters might be regarded as pertaining to applications of the subject. Thus Chapter IV, Calculus of variations and metric geometry, is devoted to a resumé of the author's applications of metric methods to obtain very general existence theorems, and to his use of such methods in the theory of curvilinear integrals. Chapter V, A general theory of length, discusses material introduced by the author in Fund. Math. **36**, 109–118 (1949) [these Rev. **12**, 15]. A generalization of normed linear space is obtained by assuming there is defined over a linear space a real-valued functional satisfying only the condition of positive homogeneity. Chapter VI, Generalized vector spaces, sketches the development of such spaces given by the author in Canadian J. Math. **1**, 94–104 (1949) and C. R. Acad. Sci. Paris **232**, 2176–2178 (1951) [these Rev. **10**, 306; **13**, 44]. The beginnings of an application of metric

methods to statistics and probability were sketched by the author in several short notes [Proc. Nat. Acad. Sci. U. S. A. **28**, 535–537 (1942); **37**, 178–180, 226–229 (1951); C. R. Acad. Sci. Paris **232**, 2001–2003 (1951); these Rev. **4**, 163; **13**, 51]. The final chapter (four pages) discusses this work. Reviewer's comments. 1. The euclidean imbedding theorems given in Chapter I are the original ones established by the author in 1928. Considerably better theorems have been available for many years [see, e.g., Blumenthal, Bull. Amer. Math. Soc. **49**, 321–338 (1943); Theory and applications of distance geometry, Oxford, 1953, Chap. IV; these Rev. **4**, 251; **14**, 1009]. 2. The congruent imbedding in Hilbert space of metric transforms of its subsets by $f(x)=x^{1/2}$, ascribed to W. A. Wilson (p. 13) is, in fact, due to I. J. Schoenberg [Ann. of Math. (2) **38**, 787–793 (1937)]. Wilson established this only for the transform of the unit segment [Amer. J. Math. **57**, 62–68 (1935)]. 3. The brief (and none too clear) sketch of a proof of the n-lattice theorem for euclidean arcs given by Alt and Beer [Ergebn. Math. Kolloq. **6**, 7 (1935)] is mentioned on page 19, but no reference is made to Schoenberg's elegant proof of this theorem for arcs of any semi-metric space with continuous distances. 4. The development of theorems reflecting the metric essence of the Frenet formulas of curve theory (expressed as desirable on p. 20) has been accomplished in the Missouri dissertation of J. W. Gaddum [Amer. J. Math. **75**, 30–42 (1953); these Rev. **14**, 581]. *L. M. Blumenthal* (Columbia, Mo.).

Convex Domains, Extremal Problems, Integral Geometry

Fejes Tóth, L. Kreisausfüllungen der hyperbolischen Ebene. Acta Math. Acad. Sci. Hungar. **4**, 103–110 (1953). (Russian summary)

The hyperbolic plane admits a tessellation $\{p, 3\}$, consisting of regular p-gons, three at each vertex, for every integer $p>6$ [V. Schlegel, Nova Acta Leop.-Carol. Deutsch. Akad. Naturforscher **44**, 337–459 (1883), especially p. 360]. Each p-gon has angle $2\pi/3$, in-radius ψ and circum-radius χ, where $\cosh\psi=\frac{1}{2}\csc\pi/p$, and $\cosh\chi=(\frac{1}{3})^{\frac{1}{2}}\cot\pi/p$. Thus its area is $(p-6)\pi/3$, the area of its in-circle is

$$2\pi(\cosh\psi-1)=\pi(\csc(\pi/p)-2),$$

and the in-circles of all the p-gons form a packing whose density is $3(\csc(\pi/p)-2)/(p-6)$. This increases with p, and its limiting value $3/\pi$ is attained by the inscribed horocycles of the faces of $\{\infty, 3\}$ [H. S. M. Coxeter and G. J. Whitrow, Proc. Roy. Soc. London. Ser. A. **201**, 417–437 (1950), p. 425; these Rev. **12**, 866].

In this beautifully illustrated paper, the author proves rigorously that every packing of proper circles has density less than $3/\pi$. *H. S. M. Coxeter* (Toronto, Ont.).

Fejes Tóth, L. Kreisüberdeckungen der hyperbolischen Ebene. Acta Math. Acad. Sci. Hungar. **4**, 111–114 (1953). (Russian summary)

Each p-gon of $\{p, 3\}$ [see the preceding review] has a circum-circle of area $2\pi(\cosh\chi-1)=2\pi 3^{-\frac{1}{2}}(\cot(\pi/p)-3^{\frac{1}{2}})$. Thus the circum-circles of all the p-gons form a covering of the hyperbolic plane, of density $2\cdot 3^{\frac{1}{2}}(\cot(\pi/p)-3^{\frac{1}{2}})/(p-6)$. This decreases as p increases, and its limiting value $2\cdot 3^{\frac{1}{2}}/\pi$ is attained by the circumscribed horocycles of the faces of $\{\infty, 3\}$.

The author proves that every covering by proper circles has density greater than $2\cdot 3^{\frac{1}{2}}/\pi$. *H. S. M. Coxeter.*

Schoenberg, I. J. On a theorem of Kirzbraun and Valentine. Amer. Math. Monthly **60**, 620–622 (1953).

The author gives a simple and elementary proof of the following theorem contained in results of M. D. Kirszbraun [Fund. Math. **22**, 77–108 (1934)], F. A. Valentine [Bull. Amer. Math. Soc. **49**, 100–108 (1943); Amer. J. Math. **67**, 83–93 (1945); these Rev. **4**, 269; **6**, 203], and E. J. Mickle [Bull. Amer. Math. Soc. **55**, 160–164 (1949); these Rev. **10**, 691]: Let $x_i, x_i', i=1, \cdots, m$, be points in euclidean n-space such that their distances satisfy $|x_i'-x_j'|\leqq|x_i-x_j|$. Then, given any point p, there exists a point p' satisfying $|p'-x_i'|\leqq|p-x_i|$. An equivalent formulation is: If m closed spheres have a common point, then they will also have a common point after being displaced such that none of the distances between their centres is increased.

W. Fenchel (Copenhagen).

Rogers, C. A. The volume of a polyhedron inscribed in a sphere. J. London Math. Soc. **28**, 410–416 (1953).

Let S be the n-dimensional unit sphere centred at the origin O in Euclidean space, and μ the Lebesgue n-dimensional measure function. The following theorem is proved: Let $\Pi\subseteq S$ be a convex polyhedron with N faces. Then

$$\frac{\mu(S)}{\mu(\Pi)}\geqq\int_0^1\frac{dt}{\{1-\bar{C}^{\delta}(1-t^2)\}^{\frac{1}{2}n}},$$

where $\delta=2/(n-1)$ and $\bar{C}=n\Gamma(\frac{1}{2}+\frac{1}{2}n)\mu(\Pi)/N\pi^{(n-1)/2}$. This result was obtained by Bambah and Davenport [same J. **27**, 224–229 (1952); these Rev. **14**, 787] for the simpler case when the foot of the perpendicular from O to a face of Π always falls inside that face. Here it is established in the general case with the aid of a general theorem on convex sets due to the author [ibid. **28**, 293–297 (1953); these Rev. **14**, 965]. The result can be used to obtain a lower bound for the covering constant for a general covering of space by n-dimensional spheres when an upper bound for N is known, since, if $N^{1/n}\to\lambda$ as $n\to\infty$, the upper limit of $\mu(\Pi)\mu(S)$ does not exceed $1-\lambda^{-2}$. *R. A. Rankin.*

Smosarski, W. Surface des tétraèdres de volume maximum inscrits dans un ellipsoide. Bull. Soc. Amis Sci. Poznan. Sér. B. **10** (1949), 40–49 (1950).

Les tétraèdres de volume maximum inscrits dans un ellipsoïde différent par leur surface. Ils ont la propriété remarquable que la somme des carrés de leurs faces est constante. C'est à l'aide de cette propriété qu'on prouve aisément que le tétraèdre de surface maximum est le tétraèdre dont les faces sont équivalentes.

Résumé de l'auteur.

Grotemeyer, Karl-Peter. Eine kennzeichnende Eigenschaft der Kugel. Arch. Math. **4**, 230–233 (1953).

Für geschlossene Flächen des gewöhnlichen Raumes wird die Formel

$$\int\int(H^2-K)P\,do=\frac{1}{4}\int\int g^{ik}H_{ik}(\mathfrak{x}^2)\,do$$

bewiesen. Hier bedeuten: do das Flächenelement, g^{ik} die kontravarianten Komponenten des ersten Fundamentaltensors, P den Abstand der Tangentialebene vom Nullpunkt, $\mathfrak{x}^2$ das Quadrat des radius vector, H die mittlere Krümmung, K die Gauss'sche Krümmung und $(\mathfrak{x}^2)_i$, H_{ik} die

Ableitungen von $\mathfrak{x}^2$ und H nach den Flächenparametern. Mit Hilfe dieser Formel ergeben sich leicht die beiden folgenden Charakterisierungen der Kugel: Eine geschlossene Fläche, von der keine Tangentialebene durch den Nullpunkt geht und bei der H eine monoton nicht wachsende Funktion von $\mathfrak{x}^2$ ist, ist eine Kugel. Eine geschlossene konvexe Fläche, bei der H eine monoton nicht wachsende Funktion von P ist, ist eine Kugel. Das Beispiel des Rotationsellipsoids zeigt, dass "nicht wachsend" hier nicht durch "nicht abnehmend" ersetzt werden darf. *W. Fenchel* (Kopenhagen).

Müller, Hans Robert. Über Momente ersten und zweiten Grades in der Integralgeometrie. Rend. Circ. Mat. Palermo (2) **2**, 119–140 (1953).

Let K be a convex curve of class C^2 in a cartesian (x_1, x_2)-plane and B the domain bounded by K. Denote the first moments of B and K with a uniform mass distribution of density 1 with respect to the x_i-axis by $F_i(B)$ and $U_i(K)$, the first moment of K with a mass distribution of density equal to the curvature by $C_i(K)$, by K^h the curve parallel to K at distance h, and by B^h the domain bounded by K^h. Then

$$F_i(B^h) = F_i(B) + hU_i(K) + \tfrac{1}{2}h^2C_i(K),$$
$$U_i(K^h) = U_i(K) + hC_i(K),$$

and, of course, $C_i(K^h) = C_i(K)$. Similar formulae are derived for the second moments. With analogous notations the following is an example:

$$U_{ij}(K^h) = U_{ij}(K) + h[C_{ij}(K) + 2\delta_{ij}F] + h^2[T_{ij} + \delta_{ij}U] + h^3\delta_{ij}\pi;$$

here U is the length of K and F the area of B, whereas T_{ij} is the second moment of the tangent image of K with a mass distribution whose density equals the radius of curvature of K at the corresponding point. These results are extended to interior parallel curves. Finally the principal cinematic formulae are derived for the first and second moments. If B' is a fixed and B a variable convex domain, then

$$\int C_i(B \cap B')\dot{B} = 2\pi\{2\pi F_i(B') + UU_i(B') + FC_i(B')\}.$$

A corresponding formula is obtained by replacing i by ij. *H. Busemann* (Los Angeles, Calif.).

Algebraic Geometry

Krygowski, Z. Sur une certaine famille de surfaces transcendantes. Bull. Soc. Amis Sci. Poznań. Sér. B. **12**, 107–111 (1953). (Russian summary)

L'auteur s'occupe de la généralisation de la surface algébrique du quatrième ordre, nommée par les architectes "arrière-voussure Saint Antoine", qui possède des propriétés semblables, a surtout aussi quatre points coniques et que l'on peut définir par l'équation:

$$\frac{z^2}{c^2} = \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{x^2y^2}{a^2b^2}\right) \cdot e^{\frac{h^2x^2y^2}{a^2b^2}}$$

qui constitue la généralisation de l'équation étudiée par le mathématicien français Rouché. *Résumé de l'auteur.*

Wylie, C. R., Jr. The existence of line involutions of order greater than three possessing a linear complex of invariant lines. Proc. Amer. Math. Soc. **4**, 807–809 (1953).

In this note the author, working exclusively on the non-singular V_4^3 in S_3 into whose points the lines of S_3 are mapped in a 1:1 way by the well known interpretation of the Plücker coordinates of a line in S_3 as point coordinates in S_5, establishes the existence of line involutions of order greater than three and whose invariant lines form a linear complex. *M. Piazzolla Beloch* (Ferrara).

Godeaux, L. L'involution de Geiser. Bull. Soc. Math. Belgique **1952**, 4–7 (1953).

Nollet, Louis. Quelques propriétés nouvelles des courbes tracées sur une surface algébrique. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8° **28**, no. 3, 40 pp. (1953).

L'auteur se propose ici d'introduire quelques notions nouvelles se rapportant à la théorie des courbes algébriques tracées sur une surface algébrique, en vue d'en faire application, dans un travail ultérieur, au problème de la classification des surfaces algébriques. Après un rappel rapide de notions connues sur la définition des variétés algébriques et sur les multiplicités d'intersection et l'équivalence arithmétique (§1), l'auteur définit au §2 les courbes positives d'une surface algébrique F: on appelle comme ça toute courbe C de F telle que $[C, X] \geq 0$ pour toute courbe irréductible X. Pour une courbe réductible C, l'auteur donne le nom de indice de connexion de C à la plus petite valeur m_C du nombre $[X, C-X]$, où X est une partie quelconque de C; l'on dit que C est connexe si $m_C > 0$, préconnexe si $m_C \geq 0$. On démontre alors que toute courbe positive effective est irréductible ou préconnexe.

Le but essentiel du §3 est le théorème de connexion, c'est-à-dire: une courbe positive effective de degré positif est irréductible ou connexe; ce théorème est précédé par quelques propriétés générales des courbes de degré positif. Comme application, si l'on appelle courbe presque irréductible une courbe effective appartenant à un système algébrique irréductible dont un autre élément soit irréductible, on trouve que toute courbe presque irréductible est positive, irréductible ou préconnexe, et si son degré n'est pas nul elle est irréductible ou connexe. Et si l'on appelle courbe pseudo-irréductible une courbe C telle que hC, pour un choix convenable de l'entier h, soit linéairement équivalente à une courbe irréductible, on trouve qu'une courbe pseudo-irréductible est irréductible ou bien préconnexe et même connexe quand son degré est positif.

Le §4 s'occupe en premier lieu des courbes fondamentales des systèmes algébriques de courbes; la courbe C, fondamentale pour un système S, de dimension $r > 0$, de courbes généralement irréductibles, est dite v-valente si les courbes de S contenant C sont ∞^{r-v}; on peut donner alors une condition pour qu'une courbe réductible C soit monovalente pour S. On trouve encore dans ce §4 des critères de régularité pour des systèmes linéaires B-adjoint préalablement définis. Le §5, enfin, s'occupe encore des courbes de degré positif; ensuite des courbes fortement positives, c'est-à-dire des courbes réductible $C = \sum C_i$ telles que $[C, C_i] > 0$ pour toute valeur de i; et des courbes strictement positives, c'est-à-dire des courbes C telles que $[C, X] > 0$ pour toute courbe irréductible X. On trouve que toute courbe de F est la différence de deux courbes fortement positives et de deux courbes effectives strictement positives. On trouve enfin que si F est irréductible, sans points multiples, et douée d'une courbe a-canonique arithmétiquement effective ou vide, on peut,

d'une infinité de manières, associer à toute courbe C de F un entier positif h tel que hC soit linéairement équivalente à la différence de deux courbes irréductibles. Toutes ces notions et ces théorèmes se rattachent aux recherches de F. Enriques et de A. Franchetta sur les courbes réductibles et sur le principe de dégénération. *E. G. Togliatti* (Gênes).

***Maroni, Arturo. Sui moduli delle curve algebriche.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 389–397. Casa Editrice Perrella, Roma, 1953.

A result of Permutti [Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **9**, 54–58 (1950); these Rev. **12**, 354] is extended to the following. If h and k $(h \geqq k > 2)$ are the minimal orders of two linear series g_h^1 and g_k^1 on an algebraic curve C of genus p, no group of g_h^1 has $k-1$ points in common with a group of g_k^1, and $p \geqq \frac{1}{2}hk - 1$ for h even or $p \geqq \frac{1}{2}(h-1)k$ for h odd, then there is not an infinite number of linear series of order h and dimension 1 on C and the number N of moduli of C is given by $N = 2h + 2k + p - 7 - \sum(s_i - 2)$, where the sum extends over all groups of s_i $(s_i > 1)$ points common to a group of g_h^1 and a group of g_k^1. When a group of g_h^1 does have $k-1$ points in common with a group of g_k^1, there are ∞^1 one-dimensional series of order h and $N' = 2h + k + p - 5$. *G. B. Huff* (Athens, Ga.).

Vesentini, Edoardo. Dimostrazione intrinseca del teorema di Riemann-Roch sopra una curva. Pont. Acad. Sci. Acta **15**, 137–152 (1953). (Latin summary)

The genus p of an algebraic curve can be defined as $\max\{n-r\}$ for all complete series g_n^r on C. Once it is proved that $n-r$ is bounded from above (and is non-negative) it is easy to show that there exist non-special series of any order $n \geqq p$ and that for any special series g_n^r one has $n \leqq 2p-2$ and $r \leqq p-1$. [This is essentially the procedure followed by Severi in his "metodo rapido" [Trattato di geometria algebrica, v. 1, parte 1, Zanichelli, Bologna, 1926, pp. 929–938] and re-elaborated by him in his recent note [Pont. Acad. Sci. Acta **14**, 143–152 (1951); these Rev. **14**, 80].] The reviewer has shown [Amer. J. Math. **58**, 1–14 (1936)] that at that stage it is possible to give an "intrinsic" proof of the Riemann-Roch theorem (i.e., a proof that belongs exclusively to the geometry on the curve C) provided it is known that there exists at least one special series of maximum order $2p-2$ (and hence necessarily of dimension $p-1$). [A short proof of this fact, more precisely, an immediate derivation of the well-known "reduction theorem" relative to the given series g_{2p-1}^{p-2}, is given by Severi in the concluding section of the present paper.] The author presents an intrinsic algebro-geometric proof of existence of such a series. His proof presupposes a knowledge of the Cayley-Brill formula for the number of fixed points of an algebraic correspondence with valence. As Severi has shown, the Schubert formula is an "intrinsic" consequence of the formula of Cayley-Brill. The author needs these two formulas for the following purposes: (1) to establish the fact that p is also the (topological) genus of the Riemann surface of C (it is this latter genus that appears in the Cayley-Brill formula); (2) to show that a g_{p+2}^2 has $p(p-1)/2$ neutral pairs. Once (2) is established, the existence of a g_{2p-2}^{p-1} is immediate. [The g_{p+2}^2 defines a plane curve C' of order $p+2$, and it seems to the reviewer that the proof which the author gives is not complete unless the curve C' has precisely $p(p-1)/2$ distinct double points (and not merely its equivalent under the form of higher singularities). It would seem then that the proof still depends on the reduction of singularities, or, more precisely, on establishing the fact that a "generic" g_{p+2}^2 has precisely $p(p-1)/2$ neutral pairs.] *O. Zariski* (Cambridge, Mass.).

Vesentini, Edoardo. Sulle molteplicità effettive delle curve polari. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 209–212 (1953).

In a recent paper [same Rend. (8) **13**, 234–238 (1952); these Rev. **15**, 58], Campedelli referred to certain arguments (about singularities of plane curves) which are now admitted to have depended on erroneous assumptions about the behavior of first polars, and he offered suggestions as to their rehabilitation. The present note, based on the author's study of polar curves and the proper interpretation of "scaricamento" and "scorrimento" [Ann. Mat. Pura Appl. (4) **34**, 219–245 (1953); these Rev. **15**, 58] contains detailed criticisms of Campedelli's suggestions and offers counter-examples to some of his assertions. *J. G. Semple.*

Dedò, Modesto. Una classica superficie del quarto ordine ed eleganti questioni ad essa collegate. Period. Mat. (4) **31**, 104–128, 176–185, 207–228 (1953)

This is a succinct and highly readable account (with four excellent diagrams) of desmic surfaces, i.e., of the irreducible quartic surfaces that can be represented by equations of the type $\{x^2, y^2; z^2, t^2\} = k$, where k is a constant and the left-hand side denotes the cross ratio of the pairs x^2, y^2 and z^2, t^2. The main headings are: (i) the general desmic surface, 16 lines, 12 nodes, and 16 conics; (ii) desmic tetrahedra; (iii) the self-collineation group G_{24} of the general desmic surface; (iv) the harmonic surface and the group G_{192}; (v) the equianharmonic surface and the group G_{288}; (vi) general classification of all quartic surfaces (without double curves) that possess inflexional conic nodes, the surface with the maximum number 12 of such nodes being desmic, and that with 10 such nodes being the Hessian of a diagonal cubic surface; (vii) the desmic surface as the evolute of a quadric. *J. G. Semple* (London).

Alguneid, A. R. Complete quadric primals in four-dimensional space. Proc. Math. Phys. Soc. Egypt **4** (1952), no. 4, 93–104 (1953).

This paper is a more or less straightforward extension to quadric primals in S_4 of the reviewer's first note on complete quadrics in S_3 [J. London Math. Soc. **23**, 258–267 (1948); these Rev. **10**, 472], with considerably more complicated algebraic computation. A complete quadric in S_4 has to be precisely specified in each of four ways, as a locus Q_0, as a complex of tangent lines Q_1, as a complex of tangent planes Q_2, and as an envelope of primes Q_3. If Σ is the 4-way space representing (to within scalar multiples) arbitrary tetrads of symmetric matrices of orders 5, 10, 10, 5, respectively, a generic complete quadric is defined unambiguously by the point of Σ which represents a tetrad of which the first member (the matrix of Q_0) has indeterminate elements, while the second, third, and fourth are the first and second components and the adjoint of the first. The totality of complete quadrics of S_4 is defined by the 4-way irreducible manifold W_{14} in Σ with the point of Σ in question as its generic point. The author finds what are, in effect, the equations of W_{14} in Σ: a set of relations (R), all linear in the separate sets of elements of the four matrices, which characterize completely the generic point of W_{14}. This leads him, by specialization, to an identification of 15 types of degenerate complete quadrics in S_4. When W_{14} is converted, in the usual way,

into a one-way manifold Ω_{14} on a Segre variety (a model in the ordinary sense of the complete quadrics of S_4) it is shown that Ω_{14} and its 15 degeneration submanifolds are all non-singular, and that four of these latter intersect simply, by pairs, by triads, and simultaneously, in the remaining eleven. *J. G. Semple* (London).

Vaccaro, Giuseppe. Esame di singolarità superficiali. II. Massimo numero di punti $(n-2)$-pli inflessionali di una superficie algebrica d'ordine n. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 203–209 (1953).

In this second note, which runs on from the first [same Rend. (8) **13**, 373–378 (1952); these Rev. **15**, 59], the author investigates surfaces F^n of S_3, free of $(n-2)$-fold lines, with as many inflexional $(n-2)$-fold points as possible.

If three such points of F^n were collinear, F^n would have to pass $(n-3)$-fold through the line containing them. So he considers first the type of F^n which has the maximum number of inflexional $(n-2)$-fold points of which no two are joined by an $(n-3)$-fold line. [This last proviso is expressed as "without acquiring $(n-3)$-fold lines", which is not what he intends.] The harmonic plane of each of the points must then pass through the others, so that there cannot be more than four of them; and it is shown that this maximum is attained only for $n=4, 6$ and for surfaces whose equations can be reduced to the forms

$$\sum a_{ij}x_i^2x_j^2=0 \quad (i, j=0, \cdots, 3; i\neq j),$$
$$\sum a_{ijk}x_i^2x_j^2x_k^2=0 \quad (i, j, k=0, \cdots, 3; i\neq j\neq k).$$

In either case, in a suitable sense, the inflexional character of three of the multiple points implies that of the fourth.

The more general case, in which collinear triads of three of the points (embedded in $(n-3)$-fold lines) are admitted, is more complicated. For $n=4$, the final stage is the well-known desmic surface with 12 inflexional nodes lying by threes on 16 lines of the surface. For $n>6$, F^n can have at most three inflexional $(n-2)$-fold points, and these will be collinear. An irreducible F^6 with 3 collinear inflexional 4-fold points can have no others. An F^5 (with no triple line) can have at most 4 inflexional triple points of which three will be collinear and the fourth will lie at the intersection of the harmonic planes of the former. *J. G. Semple.*

Andreotti, A. Recherches sur les surfaces irrégulières. Acad. Roy. Belgique Cl. Sci. Mém. Coll. in 8° **27**, no. 4, 56 pp. (1952).

This paper makes a notable contribution to the classification of irregular surfaces. The surfaces considered are those with irregularity $q\geqq 3$ which do not possess an irrational pencil of genus q. Such a surface F is then mapped, simply or multiply, on a surface Φ of the same irregularity lying on the Picard variety. The problem is essentially one of classifying surfaces contained in a Picard variety of dimension q. The author's principal tool is the theory of ideals of theta functions derived from a given Riemann matrix. It is shown that the Hilbert basis theorem applies to such an ideal, and the author derives a postulation formula and results on chains of syzygies analogous to known results for polynomial ideals. When $q=3$, a surface on the Picard variety is given by the vanishing of a single theta function; the characters of this surface are determined. When $q>3$ the situation is more complex; a theory arises which has close analogies with Gaeta's theory of space curves of finite residual. It is shown that, for $q=4$, surfaces of irregularity 4 and residual ρ fall into families depending on $2\rho+2$ integral parameters which may take arbitrarily large values. For $q>4$ the situation is still more complex.

A preliminary chapter deals with the inversion problem of Severi, and related topics (and provides incidentally a new transcendental proof of the theorem of the "completeness of the characteristic series"). It is shown that if F is represented ν-ply on its image surface Φ on the Picard variety, then the "series of irregularity", consisting of the sets of stationary points of simple integrals of the first kind, is compounded of the involution of sets of ν points on F which map the points of Φ. [See also the following review.] *J. A. Todd* (Cambridge, England).

Andreotti, A. Recherches sur les surfaces algébriques irrégulières. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8° **27**, no. 7, 36 pp. (1952).

This paper consists of three chapters, which are largely independent. I. Associated with an algebraic surface F of irregularity $q\geqq 1$ we can associate two Picard varieties [Andreotti, Rend. Accad. Naz. dei XL (4) **2**, 129–137 (1951); these Rev. **14**, 404; Igusa, Amer. J. Math. **74**, 1–22 (1952); these Rev. **13**, 680]. The author proves that these are symmetrically related, either of them being the image of the continuous ∞^q systems of hypersurfaces contained in the other. Applications are made, in particular, to prove a theorem of Torelli [Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) **22**, 2° semestre, 98–103 (1913)] which states that two algebraic curves which, for a suitable choice of retrosections and normal integrals of the first kind, have the same period matrix are birationally equivalent, and makes an extension to real curves.

II. It is proved that every surface of irregularity 2 either contains a pencil of genus 2, or else is elliptic or hyperelliptic.

III. Let F be a surface of irregularity $q\geqq 3$, not possessing a pencil of genus q. Let Φ be the image surface on the Picard variety, and suppose that Φ does not possess superlinear sheets. Then the correspondence between F and Φ is birational unless 1) Φ has torsion, and the order of the torsion group of Φ exceeds the order of the torsion group of F, ro else 2) F and Φ are elliptic surfaces. It is also proved that any algebraic surface with torsion is the image of an involution, free from coincidences, on a surface free from torsion and having the same Picard variety. *J. A. Todd.*

***Segre, Beniamino. Geometria algebrica ed aritmetica.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. I, pp. 88–98. Casa Editrice Perrella, Roma, 1953.

This communication constituted the author's preliminary announcement of his objections to a paper by Derwidué [Math. Ann. **123**, 302–330 (1951); these Rev. **13**, 67] on the resolution problem and of the contents of his then forthcoming memoir on the same problem [Ann. Mat. Pura Appl. (4) **33**, 5–48 (1952); these Rev. **14**, 683]. The reviewer of this last memoir wishes to take this opportunity to repair two omissions. In the first place, attention might well have been drawn more sharply to Segre's notable observation that the first polar of a general point of S_r with respect to a primal V_{r-1} does not always contain (to any positive multiplicity) every multiple point of V_{r-1} that lies in a neighborhood (higher than the first) of an actual multiple point of V_{r-1}. The plane curve $y^{12}+x^{113}=0$ provides a case in point, as Segre pointed out; and he gave further examples. In the second place, it has been pointed out to the reviewer that one of Segre's main objectives in the memoir in question—a proof that any algebraic surface in a non-singular V_3

can be desingularized by monoidal transformations of the V_s—had already been attained in 1944 by Zariski in the course of his paper on the reduction of singularities of three-dimensional varieties [Ann. of Math. (2) **45**, 472–542 (1944); these Rev. **6**, 102]. *J. G. Semple* (London).

Northcott, D. G. Some results concerning the local analytic branches of an algebraic variety. Proc. Cambridge Philos. Soc. **49**, 386–396 (1953).

Let V be an irreducible r-dimensional algebraic variety in the affine m-dimensional space S over a field k. Elaborating some observations of Zariski [Ann. of Math. (2) **49**, 352–361 (1948); these Rev. **9**, 460], the author proves that if V is analytically reducible at a point P, then there is a component of the singular manifold of V of dimension at least $2r-m$ passing through P. It follows that a surface in 3-space cannot be analytically reducible at an isolated singular point; on the other hand, an example is given to illustrate this possibility for a surface in 4-space. Let Q be the quotient ring of P on S, let $\bar{Q}$ be its completion, let $\mathfrak{p}$ be the prime ideal of V in Q. It is known that $\bar{Q}\mathfrak{p}=\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_h$ where the $\mathfrak{p}_i$ are prime ideals, and the author associates a "branch" of V at P with each $\mathfrak{p}_i$, calling the branch simple if the local ring $\bar{Q}/\mathfrak{p}_i$ is regular. Consider the special case where V has a general point $(\phi_1(\lambda), \cdots, \phi_m(\lambda))$, where the $\phi_i(\lambda)=\phi_i(\lambda_1, \cdots, \lambda_r)$ are rational functions over k of r independent variables. Corresponding to any elements $a_1, \cdots, a_r$ of k such that the $\phi_i(\lambda)$ are defined and finite and such that the Jacobian matrix $\|\partial\phi_i/\partial\lambda_j\|$ has rank r at $(a_1, \cdots, a_r)$, V has a simple branch at $P=(\phi_1(a), \cdots, \phi_m(a))$.
I. S. Cohen (Cambridge, Mass.).

Samuel, Pierre. Some remarks on Lüroth's theorem. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **27**, 223–224 (1953).

Let Ω be an extension of an infinite field k, let K' and K'' be two subextensions, let $(t)=(t_1, \cdots, t_r)$ be a finite set of quantities of Ω such that $K'(t)=K''(t)$. If $t_1, \cdots, t_r$ are algebraically independent over K' and over K'', then K' and K'' are k-isomorphic. If k is algebraically closed, this conclusion holds more generally if it is assumed merely that $k(t)$ is linearly disjoint from K' and from K'' over k. A simple consequence of the first result is the following generalization of the theorem of Lüroth: If $t_1, \cdots, t_n$ are algebraically independent over the infinite field k, and if K is a subextension of $k(t_1, \cdots, t_n)$ of transcendence degree 1 over k, then K is a simple extension of k. This has been proved by Igusa [same Mem. **26**, 251–253 (1951); these Rev. **14**, 81] using deep results in the theory of algebraic curves; the present proof is completely elementary. [Reviewer's note: In the proof of Theorem 1, the quantities (a) should be selected so that the $g_j(x_i', a)$ and the $f_i(\bar{x}_j'', a)$ have nonvanishing denominators.] *I. S. Cohen.*

Differential Geometry

Puckette, C. C. The curve of pursuit. Math. Gaz. **37**, 256–260 (1953).

Sevrin, A. Sur une surface réglée du quatrième ordre. Mathesis **62**, 303–314 (1953).

O'Keefe, John A. The isoparametric method of mapping one ellipsoid on another. Trans. Amer. Geophys. Union **34**, 869–875 (1953).

Künneth, Hermann, Eine Kennzeichnung lokal konvexer Kurven. J. Reine Angew. Math. **191**, 158–164 (1953).

A curve C is the continuous image of the circle in the euclidean plane. Let C^* denote the set of the points of C. If C^* is homeomorphic to a closed interval, C is called a pseudocurve. A neighborhood of a "place" on C is the image of a neighborhood of its original on the circle while a neighborhood of a point P on C^* is the connected component containing P of the intersection of C^* with a plane neighborhood of P. C is locally (punctually) convex if every place of C (point of C^*) has a neighborhood which is met not more than twice by any straight line. Thus the punctual (local) convexity of C implies its local (punctual) convexity unless C is a pseudocurve (has ramification points).

A straight line g through P is a line of local support at the place P if P has a neighborhood on C which meets g only at P and lies in one of the two half-planes bounded by g. For any point S not on C^* let $J(S)$ denote the number of lines of local support of C through S and let $\psi(S)$ be the winding number (index) of C with respect to S. The author proves: 1) Suppose C is punctally convex and not a pseudocurve. Then no straight line (of local support) meets C more than twice (once). 2) C is locally convex if and only if $J(S)+\pi^{-1}|\psi(S)|$ is both finite and independent of S. The first result generalizes a theorem by Möbius and Hjelmslev for C's which satisfy certain differentiability conditions [cf. Juel, Danske Vid. Selsk. Skr. (7) **11**, 111–167 (1914), p. 131]. *P. Scherk* (Saskatoon, Sask.).

Bouligand, Georges. Equations du premier ordre liées à un système triple orthogonal. C. R. Acad. Sci. Paris **237**, 772–774 (1953).

Dans deux notes précédentes [mêmes C. R. **236**, 2193–2195, 2462–2463 (1953); ces Rev. **15**, 34] l'auteur a rencontré l'équation (E): $pq+A(x, y, z)p+B(x, y, z)q=0$. Il la met sous forme du système (Σ): $p=(u-1)B$, $q=(1-u)A/u$, $u=u(x, y, z)$ devant satisfaire à une équation de compatibilité (λ) du premier ordre, linéaire par rapport aux dérivées u_x, u_y, u_z. U désigne l'expace euclidien (ou affine) à quatre dimensions des (x, y, z, u). Le résultat suivant est énoncé: Les intégrales paratingentes de (E) s'obtiennent en projetant sur $u=0$ les intégrales paratingentes de (λ), fournies par des variétés bidimensionnelles de U (intégrales au sens de Lie), qui sont dépourvues de tangentes parallèles à l'axe des u. *Chr. Pauc* (Nantes).

Block, H. D. A note on contact transformations. J. Math. Physics **32**, 207–208 (1953).

A contact transformation $(p_i, q_i)\to(P_i, Q_i)$ has a generating function $S(q, P)$ such that the transformation is given by $p_i=\partial S/\partial q_i$; $Q_i=\partial S/\partial P_i$ if the variables q_i and P_i are independent. The author mentions several textbooks in which it is suggested that every contact transformation for which the q_i and P_i are dependent can be generated in a similar way by another function, either $F(q, Q)$ or $F(p, Q)$ or $F(p, P)$. By an example $[Q_1=q_1, Q_2=p_2, P_1=p_1, P_2=-q_2]$ it is shown that this statement is not true. The author does not mention that this contact transformation for which $\Omega=P_2+q_2=0$ is generated by a function $S=P_1q_1$ in the following way:

$$Q_i=\frac{\partial S}{\partial P_i}+\lambda\frac{\partial\Omega}{\partial P_i};\quad p_i=\frac{\partial S}{\partial q_i}+\lambda\frac{\partial\Omega}{\partial q_i};\quad \Omega=0$$

[see E. T. Whittaker, Analytical dynamics, 4th ed., Cambridge, 1937]. *J. Haantjes* (Leiden).

Have

***Buzano, Piero. Sulla determinazione delle trasformazioni puntuali tra due piani con direzioni caratteristiche tutte coincidenti.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 282–286. Casa Editrice Perrella, Roma, 1953.

If a plane transformation has equations $x_1' = f(x_1, x_2)$, $x_2' = g(x_1, x_2)$, then f and g are solutions of a certain system of differential equations of the form

$$\frac{\partial^2 z}{\partial x_i \partial x_j} = a_{ij,1}\frac{\partial z}{\partial x_1} + a_{ij,2}\frac{\partial z}{\partial x_2} \quad (i, j = 1, 2),$$

where the $a_{ij,k}$ are functions of x_1, x_2. Conversely, any such system for which the $a_{ij,k}$ satisfy the conditions (A) of complete integrability determines the pair f, g when the initial values of these functions and of their first order derivatives are assigned, i.e., when we are given a pair of corresponding points of the transformation and the correspondence between directions at these points. The gradients m of the characteristic directions at (x_1, x_2) are given by a cubic equation in m with coefficients linear in the $a_{ij,k}$; and, for all three of them to coincide, the $a_{ij,k}$ must satisfy a set of conditions (B). The author studies the solution of the differential equations (A) for the $a_{ij,k}$, subject to (B), both generally and in special cases. The results give a corresponding analysis of the class of plane transformations whose characteristic directions are always coincident.

J. G. Semple (London).

Jonas, Hans. Zwei Klassen von Flächen, deren Bestimmung von einem Integral der Telegraphengleichung abhängt. Math. Nachr. **10**, 63–74 (1953).

A. Let $K = -1/\rho^3$ be the Gauss curvature of a surface referred to its asymptotic parameters α, β. Denote by $\mathbf{N}$ its unit normal vector and by $^*\mathbf{N}(\xi, \eta, \zeta)$ its normalized normal vector $^*\mathbf{N} =_{\text{def}} \rho^{1/2}\mathbf{N}$. The Lelieuvre formulas

$$(1) \qquad x = -\int \begin{vmatrix} d\beta & \eta_\alpha & \zeta_\alpha \\ 0 & \eta & \zeta \\ d\alpha & \eta_\beta & \zeta_\beta \end{vmatrix} \text{ cycl}$$

[cf. Hlavatý, Differentialgeometrie der Kurven und Flächen und Tensorrechnung, Noordhoff, Groningen-Batavia, 1939, p. 424; these Rev. **1**, 27] together with the Moutard equation (2) $^*\mathbf{N}_{\alpha\beta} = M\,{}^*\mathbf{N}$ and the condition (3) $1/\rho^3 = \sin^4\theta$ (where θ is the angle of $\mathbf{N}$ and the z-axis) lead easily to

$$(4) \qquad \begin{aligned} x &= 2\int (u_\alpha s d\alpha - u c d\beta) - us, \\ y &= -2\int (u_\alpha c d\alpha + u c d\beta) + uc, \\ z &= \alpha + \beta, \end{aligned}$$

where $c =_{\text{def}} \cos(\alpha - \beta)$, $s =_{\text{def}} \sin(\alpha - \beta)$, $u =_{\text{def}} \zeta$. The coefficients p and q in the general equations of a surface referred to asymptotic parameters

$$(5) \qquad x_{\alpha\alpha} = \theta_\alpha x_\alpha - p x_\beta, \quad x_{\beta\beta} = -q x_\alpha + \theta_\beta x_\beta$$

satisfy according to (4) the condition (6) $p = \theta_\alpha$, $q = \theta_\beta$ where $\theta_\alpha = \Delta_\alpha/\Delta$, $\theta_\beta = \Delta_\beta/\Delta$ and $\Delta = [^*\mathbf{N}, {}^*\mathbf{N}_\alpha, {}^*\mathbf{N}_\beta]$. These results admit different geometrical interpretations, in particular, those connected with the Laplace chain $\cdots u_\alpha, u, u_\beta, \cdots$ as well as with orthogonal systems of spheres.

B. In the case of $K = 1/4z^2 > 0$ we consider α, β as an isotherm conjugate parameter system (so that the asymptotics are given by $d\alpha^2 + d\beta^2 = 0$). Then the use of (1) and (2) leads to equations corresponding to (4) which express x, y, z by means of a solution w of the Moutard equation $w_{\alpha\alpha} + w_{\beta\beta} + w = 0$.

This review is by no means an exhaustive one. For further details one has to refer to the paper under review.

V. Hlavatý (Bloomington, Ind.).

Weitzenböck, R. W. Die Flachtangenten der ebenen Kurven vierter Ordnung. Monatsh. Math. **57**, 142–169 (1953).

Let C_4 be an irreducible algebraic curve of order four in the complex projective plane. A "flat-element" consists of a point F and a straight line which meets C_4 only at F. If F belongs to k flat-elements, it is called a k-fold flat-point. Let f denote the number of flat-elements of C_4. In general, $f = 0$. The author discusses the C_4's with $f > 0$. It is not possible to reproduce his classification. The following remarks may be useful. If a C_4 has no multiple flat-points, then either $f = 12$ or $f \leqq 6$. If there is a double (triple) flat-point, then $f = 2$, 4, 6, or 8 ($f = 3$, 4, or 5). *P. Scherk.*

Artzy, Rafael. Eigenschaften von ebenen Viergeweben allgemeiner Lage. Math. Ann. **126**, 336–342 (1953).

A four-web may be thought of as the pencils of lines in a projective plane with vertices at four points which may be called carriers. Such a web is said to be of types one, two, or three according as all four carriers are collinear, just three of the carriers are collinear, or no three of the carriers are collinear, resp. Webs of types one and two have been analyzed by K. Reidemeister [Vorlesungen über Grundlagen der Geometrie, Springer, Berlin, 1930] and by Blaschke and Bol [Geometrie der Gewebe . . . , Springer, Berlin, 1938]. The present paper analyzes webs of type three, indicating the properties which, if present, allow the introduction of a coordinate system and hence an analytic geometry into the web. It is shown that under the assumptions given an abstractly given 4-web can be embedded in a projective plane. *S. B. Jackson.*

Blaschke, Wilhelm. Osservazioni sui tessuti. Rend. Circ. Mat. Palermo (2) **2**, 36–39 (1953).

A three-web of curves in the (x, y)-plane is defined by $u_j = u_j(x, y) = \text{const.}$, $j = 1, 2, 3$, where the functional determinants $\partial(u_j, u_k)/\partial(x, y) \neq 0$ for all $j, k = 1, 2, 3$, $j \neq k$. By the elimination of x, y, it can be defined by the equation $T(u_1, u_2, u_3) = 0$. It is shown that its first invariants can be derived by the consideration of the exterior quadratic differential form

$$d\gamma = d\left\{\sum_{j=1}^{3} \frac{\partial}{\partial u_j} \log \frac{\partial T}{\partial u_j} du_j\right\}.$$

In particular, the vanishing of this form means that the web is hexagonal. The method can be extended to higher dimensions. *S. Chern* (Chicago, Ill.).

Maĭorov, V. M. Invariant characteristic of a generalized potential net. Doklady Akad. Nauk SSSR (N.S.) **90**, 965–968 (1953). (Russian)

The paper is concerned with nets of curves on a surface determined by three unit vector fields a, b and u. The main result is to the effect that any two of the following conditions imply the third: (i) the three sets of flow-lines form a hexagonal configuration; (ii) the lines of the vectors a and b form a rhombic net; (iii) the scalar field $\sin(b, u)/\sin(a, u)$ is multiply diagonal with respect to the fields a and b. [Dubnov, Trudy Sem. Vektor. Tenzor. Analizu **9**, 7–48

(1952); these Rev. **14**, 1014]. By a generalized potential net is understood one in terms of which the line element is given by $ds^2=(\partial\omega/\partial u)du^2+2fdudv+(\partial\omega/\partial v)dv^2$ where $\omega=\omega(u, v)$. The author proves that isogonal geodesic nets and rhombic nets are generalized potential nets.

M. S. Knebelman (Pullman, Wash.).

Clement, Paul A. Generalized convexity and surfaces of negative curvature. Pacific J. Math. **3**, 333–368 (1953).

An analytic surface is called a sub-K_0 surface if its Gaussian curvature K satisfies $K\leqq K_0$, where $K_0\leqq 0$ is a constant. The author generalizes results concerning the case $K_0=0$ due to E. F. Beckenbach and T. Radó [Trans. Amer. Math. Soc. **35**, 662–674 (1933)] and especially E. F. Beckenbach [Amer. Math. Monthly **55**, 285–301 (1948); these Rev. **10**, 62]. Choose a family of geodesics as parameter curves $v=$constant and their orthogonal trajectories as curves $u=$constant so that the line element of the surface takes the form $ds^2=du^2+\mu^2(u, v)dv^2$. It is shown that the surface is sub-K_0 if and only if $\mu(u, v)$ for every fixed v is a sub-K_0 function of u, that is, a generalized convex function (in the sense of Beckenbach) with respect to the family of solutions of the differential equation $h''+K_0h=0$. From this result theorems are derived which state that various functions of geometrical significance associated with the surface are sub-K_0 functions. Typical examples are the following statements. Let $a(r)$ and $l(r)$ denote the area and the length of the geodesic circle with radius r and some fixed centre P. Let further $a_0(r)$ and $l_0(r)$ denote the analogous quantities for a surface with constant curvature K_0. Then, among others, $l(r)-l_0(r)$, $a(r)-a_0(r)$, $l^2(r)-4\pi a(r)+K_0a^2(r)$ are nondecreasing sub-K_0 functions of r. The function $a^{1/2}(r)$ is sub-K_0 if and only if the surface is sub-$4K_0$. More generally, families of closed, geodesically parallel curves are considered. Some of the theorems are shown to have analogues for super-K_0 surfaces (satisfying $K\geqq K_0$). In these cases the properties in question are characteristic for sub-K_0 surfaces.

W. Fenchel (Copenhagen).

Nirenberg, Louis. The Weyl and Minkowski problems in differential geometry in the large. Comm. Pure Appl. Math. **6**, 337–394 (1953).

Weyl's problem is solved in the following form: given a line element $ds^2=Edu^2+2Fdudv+Gdv^2$ with positive Gauss curvature, where E, F, G are functions of class C^4 of analytical parameters u, v on the unit sphere, then a convex surface of class C^2 in E^3 exists with ds^2 as line element. The answer to Minkowski's is: Given a positive function $k(u)$ of class C^2 on the unit vectors in E^3, which satisfies the necessary conditions $\int_W k(u)u_i dw=0$, where W is the unit sphere, then a convex surface with $k^{-1}(u)$ as Gauss curvature exists in E^3. Uniqueness questions are not treated.

Because the great expansion of the mathematical literature makes it so hard to follow the developments, an author who treats well known problems has the duty to acquaint himself with the literature, refer the reader to the best sources, and state clearly in which respect his contribution transcends the existing results. The present paper is quite irresponsible in all these respects.

Regarding Weyl's problem, the author states that he noticed the papers of the foremost expert, Pogorelev, after completion of his work. Pogorelov assumes the existence, but not the continuity, of the fifth derivatives of E, F, G, but his methods apply to infinite surfaces and pieces of surfaces as well. All we find in the paper is that Pogorelov's assumptions are "somewhat more restrictive". Moreover, the reader is referred to sketchy Doklady notes instead of the very readable little book: Deformation of convex surfaces [Gostehizdat, Moscow-Leningrad, 1951; these Rev. **14**, 400]. The reduction to E, F, G of class C^4 is obtained by considering functions whose fourth derivatives satisfy Lipschitz conditions and then performing a limit process. This device has frequently been used by Wintner [cf., e.g., Amer. J. Math. **74**, 365–376 (1952); these Rev. **14**, 203].

As to Minkowski's problem: the necessity of

$$\int_W k(u)u_i dw=0$$

is trivial for n dimensions, if $k(u)$ is interpreted as the well-known area ratio. Here we find a complicated proof for E^3. A. D. Alexandrov and almost simultaneously Fenchel and Jessen [Danske Vid. Selsk. Math.-Fys. Medd. **16**, no. 3 (1938), where the literature is quoted] solved Minkowski's problem for general convex surfaces in E^n (the indefinite integral $\int k(u)dw$ is considered as given) by means of set functions, including uniqueness. Neither of these papers is cited. It is true that these methods yield very little regarding the differentiability properties of the solution, but for E^3 this was done by Pogorelov [Mat. Sbornik N.S. **31**(**13**), 88–103 (1952); these Rev. **14**, 679] and also by a method similar to the author's [Pogorelov, Doklady Akad. Nauk SSSR (N.S.) **83**, 361–363 (1952); these Rev. **14**, 203]. Neither of these papers is cited. The relations of the scope and hypotheses of Pogorelov's work to the author's are analogous to those in Weyl's problem. *H. Busemann.*

Minagawa, T., and Rado, T. On the infinitesimal rigidity of surfaces of revolution. Math. Z. **59**, 151–163 (1953).

Call strongly convex belt a surface of revolution (about the z-axis) generated by revolving an arc A in $y>0$ of the (x, y)-plane with the following properties. The tangents of A at its endpoints are parallel to the y-axis. A is convex and turns its concave side toward the z-axis, and has nonvanishing curvature at all interior points. Generalizing an older result of Liebmann [Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Cl. **53**, 215–234 (1901)] for the analytic case it is shown: A surface of revolution of class C' which contains a strongly convex belt of class C'' is infinitesimally rigid under deformation of class C'.

The second part generalizes a result of Rembs [Math. Z. **35**, 529–535 (1932)] in a similar way: Let B_a be an arc in $y>0$ of the (x, y)-plane which is convex, but turns its convex side towards the z-axis and has at its endpoints tangents parallel to the y-axis. Assume that B_a can be represented in the form $y=a+\rho(u)$, $z=\zeta(u)$ where u is arclength, $a>0$, $\rho(u)\geqq 0$, $\zeta'(u)>0$ and $\rho(u)$, $\zeta(u)$ are of class C''. If $\rho(u)$ and $\zeta(u)$ are kept fixed, then there is a denumerable set of values a for which B_a is not infinitesimally rigid, whereas B_a is rigid for the others. Rembs had assumed analyticity and other restrictions for $\rho(u)$, $\zeta(u)$. *H. Busemann.*

Chern, Shiing-shen. Some formulas in the theory of surfaces. Bol. Soc. Mat. Mexicana **10**, 30–40 (1953).

Let S be a closed surface in Euclidean 3-space, $\pi(P)$ the plane tangent to S at P. If P' is the foot of the perpendicular from a fixed point O to $\pi(P)$, set $p=OP'$, $d_P=PP'$, and let ρ_P equal the normal curvature of S in the direction PP'. The following identities are established:

$$(*)\qquad 2\int p^{n-1}d\Sigma+2\int p^nHd\Sigma-(n-1)\int p^{n-2}\rho_P d_P^2 d\Sigma=0,$$

$$(**)\qquad 2\int p^{n-1}Hd\Sigma+2\int p^nKd\Sigma-(n-1)\int p^{n-2}d_P^2Kd\Sigma=0,$$

where H and K are the mean and Gaussian curvatures, respectively, and the integrations are taken over all of S with respect to the area element $d\Sigma$. When $n=1$, (**) reduces to Minkowski's equation, and (*) also is independent of the choice of O. Application is made of this case ($n=1$) to furnish new proofs of Liebmann's theorems that a closed convex surface with either constant mean curvature or constant Gaussian curvature is a sphere. The methods used are similar to those in Herglotz' proof of the rigidity of convex surfaces [Abh. Math. Sem. Hansischen Univ. **15**, 127–129 (1943); these Rev. **7**, 322]. *L. W. Green.*

Hopf, Eberhard. Correction to the paper "On an inequality for minimal surfaces $z=z(x, y)$". J. Rational Mech. Anal. **2**, 801–802 (1953).

At the end of the note referred to in the title [see same J. **2**, 519–522 (1953); these Rev. **14**, 1119] the author stated a conjecture about the precise value of a certain constant occurring in the course of the discussion. The purpose of the present note is to call attention to the fact that the conjecture is false, as shown by an example exhibited by a colleague of the author. *T. Radó* (Columbus, Ohio).

Kovancov, N. I. A triorthogonal system of curves of a complex of straight lines. Doklady Akad. Nauk SSSR (N.S.) **90**, 125–128 (1953). (Russian)

In this brief note the author proves that a general line complex in three-space consists of tangents to a two-parameter family of curves. If A is the center of a ray I_3 and I_1 and I_2 are the (unit) principal normal and binormal then $dA=\omega_i I_i$ from which one obtains the three mutually orthogonal lines $\omega_1=\omega_2=0$; $\omega_2=\omega_3=0$; $\omega_3=\omega_1=0$, which are mentioned in the title. *M. S. Knebelman.*

Kovancov, N. I. Curves of a ruled complex. Ukrain Mat. Žurnal **5**, 312–324 (1953). (Russian)

In the metric geometry of a line complex one may associate a unit normal trihedral (see preceding review) for which $dA=\omega_i I_i$, $dI_i=\omega_{ik} I_k$ ($\omega_{ik}=-\omega_{ki}$, $i=1, 2, 3$) where $\omega_2=a\omega_{31}$, a being the curvature of the complex. ω_{12}, da and ω_3 are expressible linearly in terms of the remaining forms and six functions x_α of the ray I_3. In each plane there exists a curve s whose tangents are the rays of the complex and the author computes the curvature of this curve in terms of its parameter t and x_α. If the complex is ruled, then each s reduces to a point and this gives $x_1=x_2=x_3=x_4=0$, $2ax_3+x_4=0$, $a\neq 0$. If one looks for external values of this curvature, one is led to an equation of the fifth degree in t from which the author erroneously concludes that there are five principal curvatures. If the point is to be umbilical, one is led again to the above equations for x_α. There is another curve l associated with the complex whose curvature is also expressible in terms of x_α. If κ and κ_1 are the two curvatures of s and l associated with the same point, it is shown that $\kappa\kappa_1=a/t(a^2+t^2)$. *M. S. Knebelman* (Pullman, Wash.).

Decuyper, Marcel. Sur les surfaces dont les lignes asymptotiques des deux familles appartiennent à des complexes linéaires. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 688–696 (1953).

Cet article apporte une nouvelle et intéressante contribution à la question, déjà maintes fois étudiée, des surfaces dont les lignes asymptotiques appartiennent à des complexes linéaires. Si l'on considère les deux diagonals du quadrilatère de Demoulin relatif à un point quelconque d'une surface, pour que les asymptotiques de la surface appartiennent à des complexes linéaires il faut et il suffit que ces diagonales coupent les directrices de Wilczynski, relatives au même point. En outre, la division déterminée par les directrices de Wilczynski sur chaque diagonale est harmonique. L'auteur montre aussi qu'aux surfaces envisagées on peut en associer ponctuellement une infinité d'autres, de la même espèce, admettant toutes aux points homologues les mêmes quadrilatères de Demoulin. La méthode utilisée est fondée sur la considération du tétraèdre normal de E. Cartan, et met en jeu les formules fondamentales de S. Finikoff dont l'auteur a déjà fait usage dans son mémoire [J. Math. Pures Appl. (9) **26**, 15–98 (1947); ces Rev. **9**, 531]. *P. Vincensini* (Marseille).

Kruppa, Erwin. Natürliche Geometrie der Mindingschen Verbiegungen der Strahlflächen. Monatsh. Math. **55**, 340–345 (1951).

Verfasser hat [Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **157**, 143–176 (1949); diese Rev. **12**, 129] eine natürliche Geometrie der Regelflächen entwickelt, welche die Theorie der Raumkurven als Sonderfall erscheinen lässt. Sind u die Bogenlänge der Striktionslinie C, $\mathbf{e}$, $\mathbf{n}$, $\mathbf{z}$ das Rechtssystem der Einsvektoren der Erzeugenden, der Zentralnormalen und der Zentraltangente, u_1 and u_2 die Bogenlängen auf den sphärischen Bildern von $\mathbf{e}(u)$ und $\mathbf{z}(u)$, $\mathbf{t}$ der Tangentenvektor von C, dann heisst $k_1=du_1/du$ die Krümmung, $k_2=du_2/du$ die Torsion der Regelfläche φ; der Winkel zwischen $\mathbf{e}$ und $\mathbf{t}$ ist die Striktion $\sigma(u)$. Es gibt Ableitungsgleichungen für $\mathbf{e}$, $\mathbf{n}$, und $\mathbf{z}$ welche für $\sigma=0$ mit den Frenetschen Formeln der Kurventheorie übereinstimmen. Eine Mindingsche Verbiegung von φ ist eine isometrische Abbildung auf φ', wobei die Erzeugenden von φ in diejenigen von φ' übergehen. Verfasser zeigt dass man alle solche Verbiegungen von $\varphi(k^1, k^2, \sigma)$ durch die zwei, von einander unabhängigen Operationen erhält: 1) Ersetzung von k_2 durch eine beliebige andere Funktion; 2) Umkehrung des Vorzeichens von k_1 oder σ. Beispiele. Beweise bekannter und neuer Sätze. *O. Bottema* (Delft).

Finikov, S. P. Stratifiable pairs adjoined to a parabolic congruence of mutual perpendiculars. Mat. Sbornik N.S. **33(75)**, 3–12 (1953). (Russian)

There have been investigations of stratifiable congruences in which the congruence of mutual perpendiculars was pseudospherical, of the Bianchi type, isotropic, etc. This paper deals with the case that this congruence is parabolic, hence consists of tangent lines to one family of asymptotic lines of a surface. The formulas for the general case, as well as for special cases are derived in detail. Among the results for the symmetrical case we find that the surfaces with one family of asymptotic lines of constant curvature carry on the tangents to these lines a system of ∞^1 stratifiable pairs of congruences. The rays of these congruences intersect the asymptotic tangents orthogonally at points situated symmetrically with respect to the points of contact, and form with the tangent plane of the surface angles equal and of opposite sign. [See S. Finikoff, Mat. Sbornik N.S. **12(54)**, 287–314 (1943); these Rev. **6**, 19; also L. Bianchi, Atti Accad. Naz. Lincei., Rend. Cl. Sci. Fis. Mat. Nat. (5) **33**, 2° semestre, 369–377, 521–532 (1924).] *D. J. Struik.*

Berezina, L. Ya. Some theorems on bilaterally stratifiable pairs with real focal surfaces. Mat. Sbornik N.S. **33(75)**, 101–110 (1953). (Russian)

The author continues the study of bilaterally stratifiable pairs of congruences [see Doklady Akad. Nauk (N.S.) **86**,

5–6 (1952); these Rev. **14**, 405; also two articles in Latvijas PSR Zinātņu Akad. Vēstis **1951**, no. 8(49), 1317–1325; these Rev. **14**, 1120; and **1952**, no. 12(65), 145–152 (unavailable)]. From the four conditions for the system a number of theorems are derived which express geometrical properties of these pairs of congruences. *D. J. Struik* (Cambridge, Mass.).

Godeaux, Lucien. Sur la suite de Laplace associée à une surface et dont trois points appartiennent à l'hyperquadrique de Klein. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 788–797 (1953).

Etude de la suite de Laplace de l'espace réglé associée à une surface lorsque l'un de ses points, distinct de ceux qui représentent les tangentes asymptotiques à la surface, appartient à l'hyperquadrique de Klein.

Résumé de l'auteur.

Ströher, Wolfgang. Zur projektiven Geometrie der Linienelemente fünfter und vierter Ordnung. Monatsh. Math. **57**, 19–28 (1953).

As a continuation of a previous paper by the author [Monatsh. Math. **56**, 288–303 (1952); these Rev. **14**, 685], the present paper furnishes certain projective properties for the family of W-curves of the second kind having a common line element of order five or four at a given point O. It is also shown that through a line element of order six at O there is a unique real W-curve of the second kind.

C. C. Hsiung (Bethlehem, Pa.).

Kimpara, Makoto. Sur un analogue du théorème de Gauss-Bonnet en géométrie projective différentielle. J. Math. Soc. Japan **5**, 70–74 (1953).

A non-ruled surface referred to asymptotic coordinates is determined by the equations:

$$(1)\qquad x_{uu}=\theta_u x_u+\beta x_v+p_{11}x,\quad x_{vv}=\gamma x_u+\theta_v x_v+p_{22}x.$$

A reference frame is established at x whose vertices are x, $x_1=x_u+\alpha_1 x$, $x_2=x_v+\alpha_2 x$, $x_3=x_{uv}+\alpha_2 x_u+\alpha_1 x_v+\rho x$, where

$$\alpha_1=-\frac{1}{2}\left[\frac{\gamma_u}{\gamma}+\theta_u\right],$$

$$\alpha_2=-\frac{1}{2}\left[\frac{\beta_v}{\beta}+\theta_v\right],\quad \alpha_1\alpha_2-\rho=\tfrac{1}{2}(\beta\gamma+\theta_{uv}).$$

In terms of the coefficients of (1) an intrinsic linear differential form σ is defined which is invariant under homography and correlation. Its exterior derivative, Ω, is also intrinsic and invariant. Clearly $\int_C\sigma=\int\int_D\Omega$ where C is a simple curve bounding D. A geometric interpretation is given for σ, and relationships are found between this form and the quadrilateral of Demoulin and the quadric of Lie.

C. B. Allendoerfer (Seattle, Wash.).

Barner, Martin. Zur projektiven Differentialgeometrie der Komplexflächen. I. Komplexflächen als Schiebflächen. Math. Ann. **126**, 119–137 (1953).

In a three-dimensional projective space, a curve is called a complex curve if its tangents belong to a linear complex, and a surface a complex surface if the members of one family of its asymptotic curves are complex curves. In this paper, an analytic projective theory of complex surfaces is established by introducing projective motions which contain Euclidean and non-Euclidean motions as special cases. Some special projective motions, called projective displacements, are studied and used to obtain explicit parametric representations as well as existence and uniqueness theorems for complex surfaces. These projective displacements are generalizations of Clifford's displacements in elliptic, quasi-elliptic and isotropic spaces.

C. C. Hsiung.

Hlavatý, V. Embedding theory of a W_m in a W_n. Rend. Circ. Mat. Palermo (2) **1** (1952), 403–438 (1953).

The first section contains the necessary concepts involved in the geometrical investigations of a Weyl space W_m (the $g_{\mu\lambda}$ is supposed to be positive definite). The xth osculating space S^x at a point of a W_m in W_n is defined by the first, $\cdots$, $(x-1)$th covariant derivatives of the tangent vectors $T_a^\lambda=\partial_a\xi^\lambda$. The linear space in S^{x+1} normal to S^x is called the xth normal space. A set of mutually orthogonal unit vectors in the normal space together with the T_a^λ define a Frenet-connection. With respect to this connection the normal spaces are parallel. This F-connection leads to generalized Frenet formulae containing certain invariant tensors called the curvature tensors. In the fourth section an invariant parameter is introduced. With respect to this parameter the curvature tensors become invariant scalars. In section 5 the author investigates the case that the S^x are of the highest possible dimension. A definition is given of contact of order r. It turns out that it may be fully characterized by means of the curvature tensors and their derivatives. A second paper will deal with the integrability conditions of the Frenet formulae.

J. Haantjes (Leiden).

Sun, Jenning T. Frenet formulas for a subspace W_m in a Weyl space W_n. Bull. Calcutta Math. Soc. **44**, 75–85 (1952).

The main part of the paper is identical with t..e first three chapters of Hlavatý's paper [see the paper reviewed above]. Besides this, it contains a geometric interpretation of the components of the Frenet curvature tensors.

J. Haantjes (Leiden).

Hwang, Cheng-chung. On the equations of imbedding of a Riemannian space V_n immersed in a V_{n+k} of constant curvature. Acad. Sinica Science Record **5**, 23–27 (1952). (Chinese summary)

The following generalizations of results of L. L. Verbickiĭ [Trudy Sem. Vektor. Tenzor. Analizu **8**, 425–429 (1950); these Rev. **12**, 441] are derived. Suppose that the tensors $\overset{(a)}{b}_{ij}$ and $\overset{(ab)}{T}_i$ $(a, b=1, 2, \cdots, k)$ satisfy the Gauss-Codazzi equations. Then the Ricci equations will be satisfied as a necessary consequence if (1) $n\geqq 3(k-1)$, and (2) for any fixed set of indices (m, l) at least one of the matrices

$$[\overset{(a)}{M}_j, \overset{(a)}{M}_m, \overset{(a)}{M}_l],\quad j=1, 2, \cdots, n,$$

is of rank $3(k-1)$, where

$$\overset{(a)}{M}_j=\begin{Vmatrix} \overset{(1)}{b}_{1j} & \cdots & \overset{(a-1)}{b}_{1j} & \overset{(a+1)}{b}_{1j} & \cdots & \overset{(k)}{b}_{1j} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \overset{(1)}{b}_{nj} & \cdots & \overset{(a-1)}{b}_{nj} & \overset{(a+1)}{b}_{nj} & \cdots & \overset{(k)}{b}_{nj} \end{Vmatrix}.$$

Moreover, if the rank of $\overset{(k)}{b}_{ij}$ is >3, then the equations

$$\overset{(k)}{b}_{ij|m}-\overset{(k)}{b}_{im|j}=(\overset{(\mu)}{b}_{ij}\overset{(k\mu)}{T}_m-\overset{(\mu)}{b}_{im}\overset{(k\mu)}{T}_j)e_\mu,$$
$$i, j, m=1, 2, \cdots, n;\ \mu=1, \cdots k-1,$$

will be satisfied as a necessary consequence of the Gauss equations and the $n(n-1)(k-1)/2$ Codazzi equations.

D. J. Struik (Cambridge, Mass.).

Pan, T. K. Variation of congruences of curves of an orthogonal ennuple in a Riemannian space. Canadian J. Math. 5, 519–523 (1953).

Let V_n have positive definite first fundamental form; let $\lambda_h|\, h=1, \cdots, n$, be the unit tangents to n congruences of an orthogonal ennuple in V_n; let C be an arbitrary but fixed congruence of curves; and let $\lambda_h|$ be moved by local and $\lambda_h|$ by parallel displacement along C. The author then evaluates p_{hk}, the rate of change of the cosine of the angle between $\lambda_h|$ and $\lambda_k|$, and γ_{hk}, the extreme value p_{hk} can have when one considers different congruences C. Here $\gamma_{hk}=(\sum_{m=1}^{n}\gamma_{hkm}^2)^{\frac{1}{2}}$, where the γ_{hkm} are the Ricci coefficients of rotation, and γ_{hk} is called the variation of $\lambda_h|$ with respect to $\lambda_k|$. Eight theorems dealing with the variation of one congruence with respect to another are demonstrated, some stated in terms of the concepts of parallelism and equidistance of congruences [R. M. Peters, Amer. J. Math. 57, 103–111 (1935); 59, 564–574 (1937)]. Two of these theorems are: (1) In an orthogonal ennuple of normal congruences, a congruence has zero variation with respect to a second if and only if each congruence is equidistant with respect to the other or each congruence is parallel with respect to the other; (2) the variation of each congruence of an orthogonal ennuple with respect to every other congruence of the ennuple is zero if and only if the ennuple consists of congruences of geodesics, and each congruence of the ennuple is parallel along every other congruence of the ennuple. *A. Schwartz.*

Takasu, Tsurusaburo. Connection spaces in the large. I. Non-holonomic spaces with general linear connections. Yokohama Math. J. 1, 1–28 (1953).

Takasu, Tsurusaburo. Connection spaces in the large. II. Non-holonomic affine geometry. Yokohama Math. J. 1, 29–38 (1953).

Takasu, Tsurusaburo. Connection spaces in the large. III. Non-holonomic Euclidean geometry. Yokohama Math. J. 1, 39–74 (1953).

Takasu, Tsurusaburo. Connection spaces in the large. IV. Non-holonomic Laguerre geometry. Yokohama Math. J. 1, 75–77 (1953).

Takasu, Tsurusaburo. Connection spaces in the large. V. Non-holonomic conformal geometry. Yokohama Math. J. 1, 79–82 (1953).

Takasu, Tsurusaburo. Connection spaces in the large. VI. Non-holonomic Lie geometry and non-holonomic parabolic Lie geometry. Yokohama Math. J. 1, 83–87 (1953).

These six papers contain an investigation on different spaces with linear connexions in which also anholonomic coordinates are allowed. Though the title might suggest another thing, global properties in the sense now attached to this word are not considered. It is well known that tensors and other geometric objects, as for instance the object of a linear connexion, have components with respect to holonomic coordinate systems and also with respect to anholonomic systems, and it is also well known that the formulae between these latter components can be written in such a way that corresponding formulae (expressing the same geometric facts) have the same form except for some correction terms that contain the anholonomity object and that occur only where higher differentiations are involved. The author does not use this simple method and he has to duplicate all formulae in all six papers because his anholonomic formulae show some slight unnecessary differences from the holonomic ones. Moreover, he complicates things greatly by introducing "N.H. tensors", "N.H. covariant differentiation", "N.H. parallelism", "Pf. covariant differentiation" next to ordinary tensors, ordinary covariant differentiation, and ordinary parallelism instead of making use of the fact that the components arising by the use of an anholonomic coordinate system can simply be considered as anholonomic components of ordinary tensors and their ordinary derivatives. In this way a lot of theorems arise that are mere tautologies.

The first three papers do not contain much that is not known and that has not been said shorter and more clearly elsewhere. The last three papers go on in the same line and deal with what the author calls "nonholonomic" Laguerre geometry, conformal geometry, and Lie geometry.

J. A. Schouten (Epe).

Truesdell, C. Generalization of a geometrical theorem of Euler. Comment. Math. Helv. 27, 233–234 (1953).

The author reinterprets Euler's derivation of the equation of continuity in hydrodynamics in terms of affine motions. He considers the transformation $\bar{x}=(At+I)x$ of vectors in n-space, where A is an $n\times n$ matrix, I the unit matrix and t any number. The volume of a parallelotope with vertices at 0 and x_i $(i=1, 2, \cdots, n)$ being defined as $V=\det x_i$, volume transforms according to $\bar{V}=V\det(At+I)$; hence $\bar{V}/V=\sum_{i=0}^{n}I_i t^i$ where I_i is the sum of the i-rowed principal minors of A. Euler proved this result by explicit calculation for $n=2, 3$, and so is the discoverer of the secular expansion. The following formulae are obtained:

$$\frac{1}{r!V}\frac{d^r\bar{V}}{dt^r}\bigg|_{t=0}=\begin{cases}I_r, & r=1, 2, \cdots, n,\\ 0, & r\geqq n+1,\end{cases}$$

$$J_r=\frac{(-1)^r}{(r-1)!}\frac{d^r\log\bar{V}}{dt^r}\bigg|_{t=0},$$

$$\log\frac{\bar{V}}{V}=\sum_{r=1}^{\infty}\frac{(-1)^{r-1}J_r}{r}t^r,$$

where $J_r=$ sum of rth powers of proper numbers of A.

J. L. Synge (Dublin).

Yano, Kentaro, et Mogi, Isamu. Sur les variétés pseudo-kähleriennes à courbure holomorphique constante. C. R. Acad. Sci. Paris 237, 962–964 (1953).

An even-dimensional Riemannian manifold $ds^2=g_{ij}dx^i dx^j$ is called pseudo-Kaehler if there exists a skew tensor ϕ_{ij} for which $\phi_{ij,k}=0$. Also, at a point, a 2-plane is called holomorphic if it is determined by two orthonormal vectors u^i, v^i which are connected by $v^i=\phi_j{}^i u^j$, and, for such, $K\equiv -R_{ijkl}u^i v^j u^k v^l$ is a holomorphic curvature. Theorems. At a point, the holomorphic curvature is constant, $=k$, if and only if, at the point,

$$R_{ijkl}=\frac{k}{4}[(g_{jk}g_{il}-g_{ji}g_{kl})-(\phi_{jk}\phi_{il}-\phi_{ji}\phi_{kl})-2\phi_{ij}\phi_{kl}].$$

If this is so at all points, then k is the same for all points, and for any two orthonormal vectors we have

$$K=\frac{k}{4}(1+3\phi_{ij}u^i v^j),$$

so that in particular $|k|/4\leqq|K|\leqq|k|$, $Kk>0$, and for $k>0$ any two conjugate points have the same geodesic distance $2\pi/k^{1/2}$. Also, constant holomorphic curvature is equivalent with "free holomorphic mobility". *S. Bochner.*

Yano, Kentaro. On Killing vector fields in a Kaehlerian space. J. Math. Soc. Japan 5, 6–12 (1953).

In a real Riemannian space a Killing vector is defined by $\xi_{i,j}+\xi_{j,i}=0$ and the author found previously [Ann. of Math. (2) 55, 38–45 (1952); these Rev. 13, 689] that on a compact orientable space this is equivalent with

$$g^{ih}\xi^j_{,j,h}+R^i_j\xi^j=0 \quad \text{and} \quad \xi^j_{,j}=0.$$

Now, on a (non-compact) Kaehler space the author now replaces the second condition by the sharper one $\xi^\alpha_{,\alpha}=0=\xi^{\bar\alpha}_{,\bar\alpha}$ and he shows (i) that a Killing vector satisfying it is also a harmonic vector and (ii) that on a compact Kaehler space if a vector satisfies (only) this condition and if ξ^α is holomorphic in the z^β and $\xi^{\bar\alpha}$ in the $z^{\bar\beta}$, then $\xi^i=(\xi^\alpha;\xi^{\bar\alpha})$ is also a Killing vector. *S. Bochner* (Princeton, N. J.).

Westlake, W. J. Conformally Kähler manifolds. Proc. Cambridge Philos. Soc. 50, 16–19 (1954).

The author considers a space H_n of n complex dimensions with a Hermitian metric $ds^2=g_{\alpha\bar\beta}dz^\alpha dz^{\bar\beta}$ and defines a metric linear connection by $\Gamma^\alpha_{\beta\gamma}=g^{\alpha\bar\epsilon}(g_{\bar\epsilon\beta,\gamma})$, where the comma denotes partial differentiation with respect to z. The metric is Kaehlerian when and only when the torsion tensor $S^\alpha_{\beta\gamma}=\frac{1}{2}(\Gamma^\alpha_{\beta\gamma}-\Gamma^\alpha_{\gamma\beta})$ vanishes. He then considers a conformal transformation $g'_{\alpha\bar\beta}=\sigma g_{\alpha\bar\beta}$ of Hermitian metric, where σ is a real positive function of the coordinates, and derives the three following conformal tensors:

$$C^\alpha_{\beta\gamma}=S^\alpha_{\beta\gamma}+\frac{1}{n-1}(\delta^\alpha_\beta S_\gamma-\delta_\gamma{}^\alpha S_\beta),$$

$$b_{\alpha\beta}=S_{\alpha,\beta}-S_{\beta,\gamma}, \quad b_{\alpha\bar\beta}=S_{\alpha,\bar\beta}-S_{\bar\beta,\alpha},$$

where $S_\alpha=S^\epsilon{}_{\alpha\epsilon}$. If there exists, in H_n, a positive scalar function σ such that $g'_{\alpha\bar\beta}=\sigma g_{\alpha\bar\beta}$ gives a Kaehler metric, we call H_n a conformally Kaehler manifold and denote it by X_n. The author proves: A necessary and sufficient condition that H_n $(n>2)$ shall be conformally Kaehler is that the conformal tensor $C^\alpha_{\beta\gamma}$ vanishes identically; for $n=2$, the conditions are that $b_{\alpha\beta}$ and $b_{\alpha\bar\beta}$ vanish identically; and for $n=1$, the problem is trivial.

This is a local theorem; but, using a theorem of N. Coburn [Trans. Amer. Math. Soc. 50, 26–39 (1941); these Rev. 3, 19] and an argument similar to that used by N. H. Kuiper [Ann. of Math. (2) 50, 916–924 (1949); these Rev. 11, 133], he proves a global theorem: A conformally Kaehler manifold X_n, which is simply connected or whose fundamental group contains elements of finite order only, can be given a Kaehler metric. *K. Yano* (Rome).

Calabi, Eugenio, and Rosenlicht, Maxwell. Complex analytic manifolds without countable base. Proc. Amer. Math. Soc. 4, 335–340 (1953).

Etant donnée une variété analytique complexe V de dimension n et un point $P \in V$, il existe une application analytique T_P d'une variété analytique complexe V_P sur V telle que $T_P^{-1}P=U_P$ soit un espace projectif complexe de dimension $n-1$ (correspondant aux directions complexes de V en P) et qui applique biunivoquement V_P-U_P sur $V-P$. Soit W une sous-variété analytique de V, sans singularité, de dimension complexe un, et soit $W_P=T_P^{-1}W$. L'ensemble M obtenu à partir de la réunion des V_P-W_P (pour une infinité non dénombrable de points $P \in W$), en identifiant les points des variétés V_P-W_P qui correspondent au même point de $V-W$, est muni d'une structure de variété analytique complexe. Chacune des V_P-W_P induit un ouvert de M, mais aucune sous-famille dénombrable de tels ouverts ne recouvre M. Dans l'article, T_P est définie explicitement par une transformation quadratique, lorsque V est une variété algébrique, projective, sans singularité.

Pour t réel, soit E_t le plan euclidien réel (x_t, y_t), considérons la relation d'équivalence suivante entre les points des E_t: $(x_s, y_s)\sim(x_t, y_t)$ signifie: $y_s=y_t$ et $x_sy_s+s=x_ty_t+t$ si $y_s=y_t>0$; $s=t$, $x_s=x_t$ si $y_s=y_t\leqq 0$. L'ensemble S_+ des classes d'équivalence est muni d'une structure de variété analytique réelle. Il en est de même de la surface réelle S, définie à partir des E_t et de la relation d'équivalence: $(x_s, y_s)\sim(x_t, y_t)$ signifie: $y_s=y_t$ et $x_sy_s+s=x_ty_t+t$ si $s\neq t$; $x_s=x_t$ si $s=t$. Les surfaces S_+ [introduites par Prüfer, cf. T. Radó, Acta Litt. Sci. Szeged 2, 101–121 (1925)] et S n'admettent pas de base dénombrable. Soit désormais M la variété complexe obtenue à partir de S en donnant à t, x_t, y_t les valeurs complexes, elle peut être obtenue, à partir du plan (x, y) à 2 dimensions complexes, par le procédé du début appliqué à tous les points de la droite complexe $y=0$. Les variétés S_+, S et M possèdent les propriétés topologiques suivantes: S_+ n'est pas séparable, S et M sont séparables; S_+, S et M ne sont pas normales, donc pas métrisables; S_+ est contractile; S a un groupe fondamental non dénombrable; M est simplement connexe, mais son second groupe d'homotopie est non dénombrable. *P. Dolbeault* (Paris).

***Kodaira, Kunihiko. The theorem of Riemann-Roch for adjoint systems on Kählerian varieties.** Contributions to the theory of Riemann surfaces, pp. 247–264. Annals of Mathematics Studies, no. 30. Princeton University Press, Princeton, N. J., 1953. $4.00.

This paper is an account of Riemann-Roch theorems obtained by the author for adjoint systems on Kählerian varieties of arbitrary dimension.

Let M_n be a compact Kählerian variety of complex dimension n. If there exists at least one non-trivial meromorphic n-ple differential W on M_n, its divisor K defines the canonical divisor class, and the adjoint system of an arbitrary division D on M_n is the complete linear system $|K+D|$. If no canonical divisor exists, the adjoint system is defined to be the empty set. Denote the number of linearly independent λ-ple differentials of the first kind on M_n by $r_\lambda(M_n)$, and let $l=l(S)$ be the number of linearly independent $(n-1)$-ple differentials of the first kind on M_n which vanish when restricted to a given irreducible non-singular subvariety S of dimension $n-1$. The following theorem is proved [cf. Kodaira, Proc. Nat. Acad. Sci. U. S. A. 38, 522–527 (1952); these Rev. 14, 82]:

$$\dim|K+S|=r_n(M_n)-r_{n-1}(M_n)+r_{n-1}(S)+l-1.$$

The second part of the paper concerns 3-dimensional Kählerian varieties; a detailed exposition of these results is contained in the author's paper [Ann. of Math. (2) 56, 298–342 (1952); these Rev. 14, 790]. *D. C. Spencer.*

Segre, Beniamino. Dilatazioni e comportamenti associati nel campo analitico. Rend. Circ. Mat. Palermo (2) 1 (1952), 373–379 (1953).

In this paper the author investigates in the domain of analytic manifolds, concepts of dilatation and associated behavior analogous to similar concepts which he has recently investigated in the domain of algebraic geometry [Ann. Mat. Pura Appl. (4) 33, 5–48 (1952); these Rev. 14, 683]. Since the new paper is fairly self-contained, the reader is left to discover for himself how far its definitions and results can be fitted on to the purely algebraic argument of the earlier work.

Dilatation, in the present connection, is defined essentially as follows. Let U be an analytic manifold of dimension $\nu+1$ ($\nu \geqq 1$), and let T be an analytic submanifold of U, of dimension t ($0 \leqq t < \nu$). Also let O be a simple point of T, which is simple also for U; and let τ be a transformation, with T as fundamental manifold, which carries U into another $(\nu+1)$-dimensional analytic manifold U', and the point O in particular into a locus of $\infty^{\nu-t}$ points O' of U'. Suppose further, now, that for each point-pair (O, O') so arising, there exist local coordinate systems $x_0, \cdots, x_\nu$ on U and $x_0', \cdots, x_\nu'$ on U', valid in neighborhoods Δ and Δ' of O and O' respectively, such that the equations of τ in these local coordinates are precisely

$$(1) \qquad x_0 = x_0'; \quad x_0 = x_0' x_i' \quad (i = 1, \cdots, \nu - t); \qquad x_j = x_j' \quad (j = \nu - t + 1, \cdots, \nu)$$

where T has now the equations $x_0 = x_1 = \cdots = x_{\nu-t} = 0$. Under these conditions, then, τ generates, in the neighborhood of O, a dilatation of base T of U, the whole neighborhood Δ' of any one of the homologues O' of O being the transform of a thin "wedge" of the neighborhood Δ of O.

Suppose now that V, W are analytic primals (ν-dimensional manifolds) in U, given by equations $f(x) = 0$ and $g(x) = 0$ in the local coordinates at O. W is said to have "associated behavior of index r ($\geqq 1$) with V at O" if

$$g(x) \equiv a(x) f(x) + \sum_{s \leqq r} b_s(x) \omega^{(s)} f(x),$$

where $a(x)$, $b_s(x)$ are regular at O and ω^s is an arbitrary sth order derivation with respect to the x_i. Further, the associated behavior is said to be "regular" if

$$\sum b_r(0) \omega^{(r)} \phi_m(x) \not\equiv 0,$$

where $\phi_m(x)$ is the aggregate of terms of least degree m ($m > r$) in $f(x)$; and this means that when V has an m-fold point at O, W has multiplicity $m - r$ at O, i.e., the least possible for associated behavior of index r.

By systematic use of equations (1), the author obtains the following results. (I) If V passes m-fold through T and has precisely an m-fold point at O, and if W has regular associated behavior of index r with V at O, then the transforms V', W' of V, W by a dilatation of base T of U are such that W' has associated behavior of index r with V at every point O' corresponding to O. (II) Under the sam' conditions, if V' has the same multiplicity m at O' as had V at O, the associated behavior of W' with V' at O' is regular.

J. G. Semple (London).

Fulton, Curtis M. Generalized Weierstrass tensors. Tensor (N.S.) **3**, 23–25 (1953).

The author defines a tensor Δ_i^λ [the B_i^λ in Schouten and Struik, Einführung in die neueren Methoden der Differentialgeometrie, Bd II, Noordhoff, Groningen-Batavia, 1938] by which the projection of an affinor of V_m or V_n (the V_m in V_n is given by the equations $F^\alpha(x^1, \cdots, x^n) = 0$) can easily be constructed.

J. Haantjes (Leiden).

***Julia, Gaston. Cours de géométrie infinitésimale. Premier fascicule. Vecteurs et tenseurs. Théorie élémentaire.** 2ème éd. Gauthier-Villars, Paris, 1953. xv +102+i pp. 2000 francs.

This is a revision of the author's textbook written for the use of students in the Ecole Polytechnique. It covers the standard material on the algebra of vectors and the properties of curl, divergence, and gradient. In addition, the author treats the properties of cartesian tensors in rectangular and oblique coordinates. The chief innovation since the first edition is the discussion of symmetric tensors of the second order in relation to the associated quadratic forms. It is regrettable to learn from this book that in France, as well as in the United States, there has been no impact of modern theories of linear algebra and differential forms upon the teaching of vectors and tensors to university students.

C. B. Allendoerfer (Seattle, Wash.).

NUMERICAL AND GRAPHICAL METHODS

***Tables of circular and hyperbolic sines and cosines for radian arguments.** National Bureau of Standards Applied Mathematics Series, No. 36. U. S. Government Printing Office, Washington, D. C., 1953. x+407 pp. $3.00.

The first edition of these tables was published in 1939 by the Works Project Administration for the City of New York [cf. these Rev. **2**, 64]. A 2d edition was published in 1949 by the National Bureau of Standards as MT3. All reported errors in previous editions have been corrected in this edition.

Kotani, Masao, Ishiguro, Eiichi, Hijikata, Katsunori, Nakamura, Takashi, and Amemiya, Ayao. Tables of integrals useful for the calculations of molecular energies. III. J. Phys. Soc. Japan **8**, 463–476 (1953).

The first two parts of this paper appeared as Proc. Phys.-Math. Soc. Japan (3) **20**, extra no. 1 (1938); **22**, extra no. 1 (1940); these Rev. **2**, 239. The present third part adds a number of formulas, contains a list of corrections to the first two parts, and gives 7 pages of numerical tables.

A. Erdélyi (Pasadena, Calif.).

van der Hoff, B. M. E., and Benson, G. C. A method for the evaluation of some lattice sums occurring in calculations of physical properties of crystals. Canadian J. Physics **31**, 1087–1094 (1953).

The authors illustrate a method, due to Mackenzie, for the numerical evaluation of sums of inverse nth powers of distances from a given point P to all points of an infinite lattice. A cubic lattice sum such as $\Sigma'(k^2+l^2+m^2)^{-n}$, where the sum extends over all positive and negative integers k, l, m, not all zero, is evaluated in terms of gamma, theta, zeta and Bessel functions as follows. The summand is written

$$(k^2+l^2+m^2)^{-n} = \frac{1}{\Gamma(n)} \int_0^\infty t^{n-1} \exp\{-(k^2+l^2+m^2)t\}\, dt.$$

The operator Σ' is now applied to the exponential function and the theta function transformation

$$\sum_{m=-\infty}^{\infty} \exp(-m^2 t) = (\pi/t)^{1/2} \sum_{m=-\infty}^{\infty} \exp(-m^2 \pi^2 t^{-1})$$

is applied to two of the three sums. Limits are changed to 0 and ∞ with the introduction of some zeta-function terms.

This leads to integrals of the form

$$\int_0^\infty t^{n-2} \exp\{-(k^2t+\pi^2(l^2+m^2)t^{-1}\}dt.$$

These are essentially Bessel functions of imaginary argument and of the second kind and are given by

$$2(k^{-1}(l^2+m^2)^{1/2}\pi)^{n-1}K_{n-1}(2k(l^2+m^2)^{1/2}\pi).$$

A few terms of this type give the desired sum to 8 or more decimals. Examples are given in the plane to square and hexagonal lattices for points P both on and off the lattice. Formulas also include examples of the cubic lattice with and without restriction as to parity of the indices of summation which occurs in chemical physics. *D. H. Lehmer.*

***Hruška, Václav. Počet grafický a graficko-mechanický. [Graphical and graphical-mechanical computation.]** Přírodovědecké Vydavatelství, Prague, 1952. 1070 pp. 600 Kčs.

Ch. I. Introduction. Ch. II. Arithmetic and algebra. Ch. III. Functions of one variable: representation of a function on millimeter paper; logarithmic paper; scale of a function; other graph papers. Ch. IV. Nomography: intersection nomograms; alignment nomograms; nomograms with transparent sheets; anamorphosis; conclusion to nomography. Ch. V. Analysis: derivation and integration of functions; differential equations of one argument; partial differential equations. Ch. VI. Mechanical aids to graphical computation: integraphs and differentiators; integrators and harmonic analyzers; slide rules and mechanized nomograms. Ch. VII. Supplements. Bibliography. Index.

Table of contents.

Dienst, Hans-Rudolf. Untersuchungen zum Eindeutigkeitsproblem der Nomographie. Bl. Deutsch. Ges. Versicherungsmath. **1**, no. 4, 71–96 (1953).

This paper considers n-parameter families of curves under topological transformations of the plane. In a region each family is given by $f(x, y) = \text{const}$. The investigation concerns the possibility of simultaneously rendering the n f's linear by a topological transformation. For $n=3$, functions f, g, h give a nomogram if they make $F(f, g, h)$ identically zero. With suitable differentiability and region, the condition that the three families be straight lines is a differential system S. The projective group leaves the general solution of S invariant. A result limiting the other possible types of solution is found. *J. M. Thomas* (Durham, N. C.).

***Shaw, F. S. An introduction to relaxation methods.** Dover Publications, Inc., New York, N. Y., 1953. 396 pp. $5.50.

The book gives a detailed account of the solution by relaxation methods of linear simultaneous equations and ordinary and partial linear differential equations. There is an extensive bibliography (to the end of 1949). After a brief account of the original work by Southwell in connection with frameworks, the solution of linear simultaneous equations is considered. A full explanation of the process applied to a set of equations is given with all the steps in the solution, and the accuracy that can be obtained is discussed. The solution of linear ordinary differential equations is treated in a similar manner. Most of the book is devoted to the solution of linear partial differential equations of the second and fourth order. The method is first described in detail as applied to the solution of Poisson's equation $\nabla^2\phi = Z(x, y)$ over simple boundary shapes and then other topics such as curved boundaries, various boundary conditions (specified function values or derivatives) and more complicated equations are considered. The solution of eigenvalue problems are next considered and the book concludes with an outline of methods for the solution of integral equations, free boundaries and the improvement of accuracy using the method of Fox [Proc. Roy. Soc. London. Ser. A. **190**, 31–59 (1947); these Rev. **9**, 106]. Tables of coefficients for use with curved boundaries are included.

D. C. Gilles (London).

Mitchell, A. R., and Rutherford, D. E. On the theory of relaxation. Proc. Glasgow Math. Assoc. **1**, 101–110 (1953).

The authors discuss from a common mathematical viewpoint various relaxation or iterative procedures for solving systems of linear algebraic equations (with emphasis on systems arising from finite-difference approximations to differential equation problems. They adopt the magnitude of the residual vector as the norm of error in the approximation. They explain the relationship between various measures of "condition" of a matrix. They conclude with a discussion of the application of the relaxation technique to (a) an ordinary differential equation, (b) the boundary-initial-value problem for the heat equation. (Apparently examples were treated in a less direct fashion in an earlier paper by Allen and Severn.) *E. Isaacson.*

Targonszky, G. An always convergent iteration process. Acta Math. Acad. Sci. Hungar. **4**, 119–126 (1953). (Russian summary)

An iterative method is presented for the approximate solution of equations which involve continuous functions. In particular, this algorithm is applied to the calculation of the roots of polynomial equations. *E. Frank.*

***Newmark, N. M. Bounds and convergence of relaxation and iteration procedures.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 9–14. The American Society of Mechanical Engineers, New York, N. Y., 1952.

Consider a set of linear simultaneous equations, the errors r_i of an approximate solution x_i being given by $r_i = \sum_k a_{ik}x_k + b_i$. If $x_i + u_i$ is a closer approximation to the correct solution ϕ_i, the value of t can be chosen so that $x_i + tu_i$ is a still closer approximation. The choice of t may be made to determine: (i) the upper and lower bounds of some specified quantity $\theta_n = \sum_i a_{ni}\phi_i$; (ii) the best next value of $x_i + tu_i$ for an iteration procedure, in which x_i after one iteration gives $x_i + u_i$, and y_i similarly gives $y_i + v_i$; (iii) a closer solution in a formal relaxation process; a slight modification of the method of steepest decents of Temple [Proc. Roy. Soc. London. Ser. A. **169**, 476–500 (1939)]; (iv) the solution of a formal relaxation process by using orthogonal corrections. If u_i and v_i are such corrections, then their error charges r_i are mutually orthogonal, i. e., $\sum m_i p_i q_i = 0$, when the m_i are weighting factors and $p_i = \sum a_{ik}u_k$ and $q_i = \sum a_{ik}v_k$. The solution can be explained as a series of n of these orthogonal corrections, where n is is the order of the set of simultaneous equations.

D. C. Gilles (London).

Zuhovickiĭ, S. I. On best approximation in the sense of P. L. Čebyšev of a finite system of incompatible linear equations. Mat. Sbornik N.S. **33(75)**, 327–342 (1953). (Russian)

Let there be given a set of m equations in n unknowns, $Ax=b$, where A is an $m\times n$ matrix, $m>n$, b is an m-vector and x is an n-vector. The problem is to find an x for which the residual m-vector $r=Ax-b$ has the least possible value for its longest component. Though the idea is simple, it does not lend itself readily to calculation. The purpose of the paper is to present an algorithm for the actual numerical evaluation of the vector x. Two cases are considered, first that in which the condition of Haar is satisfied, namely where all n-rowed determinants in A are non-zero, and second, where this condition is not fulfilled. The author devises computational procedures for each case and illustrates each with a numerical example. *W. E. Milne.*

***Charnes, A., and Lemke, C. E. Computational problems of linear programming.** Proceedings of the Association for Computing Machinery, Pittsburgh, 1952, pp. 97–98. Richard Rimbach Associates, Pittsburgh, Pa., 1952.

***Dwyer, Paul S. Errors of matrix computations.** Simultaneous linear equations and the determination of eigenvalues, pp. 49–58. National Bureau of Standards Applied Mathematics Series, No. 29. U. S. Government Printing Office, Washington, D. C., 1953. $1.50.

As the title implies the author is concerned with those errors in matrix computation which arise in numerical calculations. He considers the notion of an approximate matrix and operations with such matrices. In particular, the solution of approximate simultaneous equations is discussed. *H. H. Goldstine* (Princeton, N. J.).

Banachiewicz, T. Les cracoviens et quelques-unes de leurs applications en géodésie. Bull. Soc. Amis Sci. Poznań. Sér. B. **10** (1949), 3–39 (1950).

Saibel, Edward, and Berger, W. J. On finding the characteristic equation of a square matrix. Math. Tables and Other Aids to Computation **7**, 228–236 (1953).

For obtaining the characteristic equation of A, Hessenberg first reduces A by a similarity transformation to a special form $-P$, which is upper triangular except for ones down the subdiagonal. The authors summarize the description given by Zurmühl [Matrizen, Springer, Berlin, 1950; these Rev. **12**, 78]. Next they point out that if only the first element of x_0 is non-null, and the characteristic equation of P is $c_0+c_1\lambda+\cdots+\lambda^n=0$, then the equations $(c_0+c_1P+\cdots+P^n)x_0=0$ in the c's have a triangular matrix and are therefore easily solved. Finally they find that the number of multiplicative operations required is of the order of n^3 as compared with n^4 required by other methods in the literature. The authors speak of the entire scheme as "fully automatic", although special cases can occur in the development of P that would appear to complicate programming for machine computation. *A. S. Householder.*

Lotkin, M. A new integration procedure. J. Math. Physics **32**, 171–179 (1953).

Main concern are quadrature formulas of the type $\int_0^h f(x)dx=h\sum_{k=-m}^{m+1}p_kf(kh)$ with constant weights p_k, especially for $m=1$, 2 and 3. These formulas are obtained by means of central differences and by means of the special quadrature formula

$$(1)\quad \int_0^h f(x)dx=\frac{h}{2}(f_0+f_1)+\frac{h^2}{10}(f_0'-f_1')+\frac{h^3}{120}(f_0''+f_1'')+R;$$

$$R=-\frac{h^7}{100\,800}f^{(vi)}(\xi);\quad 0\leqq\xi\leqq h;$$

$$f_0=f(0),\quad f_1=f(h).$$

The new formulas are identical with the well-known Stirling quadrature formulas for $m=1$ and $m=2$. The author's formula for $m=3$ is less accurate than Stirling's, as (1) does not come up to the required accuracy. The formulas of Stirling are connected with the polynomials $\prod_{k=-m}^{m+1}(x-k)$ and are described in many textbooks on the calculus of differences, especially in the book of E. Nörlund, Vorlesungen über Differenzenrechnung [Springer, Berlin, 1924]. *H. Bückner* (Schenectady, N. Y.).

Lotkin, Mark. The treatment of boundary problems by matrix methods. Amer. Math. Monthly **60**, 11–19 (1953).

Using matrix notation throughout, the author obtains a solution of the linear differential equation of the nth order $(D=d/dx)$

$$L(D)y=[a_nD^n+a_{n-1}D^{n-1}+\cdots+a_0]y=r(x),$$

satisfying the set of boundary conditions at p points $(p\leqq n,\ m\leqq n-r)$

$$M_p(D)y_p=[b_{p,m}D^m+b_{p,m-1}D^{m-1}+\cdots+b_{p,0}]y_p=\rho_p,$$

in the form $y_s=Y_0+\sum_{k=1}^{s}C_kY_k$, where Y_0 satisfies the boundary conditions, and the Y_k the set of associated homogeneous conditions $M_p(D)y_p=0$. The coefficients C_k are determined by the method of least squares, i.e. by minimising $\int_{x_0}^{x_1}[L(D)y_s-r]^2dx$. An example is given of the method applied to a second-order equation taking $s=3$ and 4. The results of the least-square determination, and also the determination of the C_k by Galerkin's method are compared with the known solution. The method is also applied to a set of simultaneous linear differential equations. *D. C. Gilles* (London).

David, F. N. A note on the evaluation of the multivariate normal integral. Biometrika **40**, 458–459 (1953).

Zadiraka, K. V. The approximate integration by S. A. Čaplygin's method of linear differential equations of the 2d order with variable coefficients. Ukrain. Mat. Žurnal **4**, 299–311 (1952). (Russian)

In 1919 Čaplygin [Collected works, vol. I, Gostehizdat, 1948, pp. 347–444; these Rev. **14**, 609] gave a method for the approximate solution of differential equations of the first order based on certain theorems about differential inequalities. The present paper extends Čaplygin's method to the case of a linear differential equation of the second order. It is necessary to consider separately the cases of oscillatory and non-oscillatory equations, and an illustrative example of each kind is worked out in the paper. *W. E. Milne* (Corvallis, Oregon).

Mounier, C. Généralités sur les principes de fonctionnement des machines mathématiques électriques analogiques. Rev. Gén. Electricité 62, 515–530 (1953).

van der Corput, J. G. Modern calculating machines. Simon Stevin 29, 203–228 (1952). (Dutch)

Ford, R. L. Differentiating circuits. Electronic Engrg. 25, 519–521 (1953).

Freudenthal, H. Integrators. Simon Stevin 29, 177–184 (1952). (Dutch)

Holecek, K. Ein Beitrag zum Maschinenrechnen: Die Berechnung vielstelliger Quotienten nach dem Aufbauverfahren. Österreich. Ing.-Arch. 7, 331–336 (1953).

***Flad, J. P. L'extraction automatique de la racine carrée. La première machine de bureau réalisant l'extraction automatique de la racine carrée.** Centre d'Information du Matériel et des Articles de Bureau, Paris, 1953. 16 pp. 300 fr.

Antonowicz, K. An integrating apparatus for the Schrödinger equation. Acta Phys. Polonica 12, 163–169 (1953). (Russian summary)

ASTRONOMY

***Lyttleton, R. A. The stability of rotating liquid masses.** Cambridge, at the University Press, 1953. viii+150 pp. $6.50.

The forms of gravitating fluid masses in steady uniform rotation and their stability have been the subject of many extensive memoirs by Darwin, Liapounoff, Jeans, Poincaré and Cartan. The present book establishes the main principles and results while referring to the original papers for the details of the calculations. The book includes an account of ellipsoidal harmonics which though brief is adequate for the purposes of the subject.

The principal results of the theory are that of Poincaré that the Jacobi ellipsoid is secularly unstable at a certain speed of rotation; that of Liapounoff and Jeans that when the speed is slightly exceeded beyond the point of secular stability there is no neighboring configuration that is secularly stable; and finally that of Cartan that when the Jacobi ellipsoids become secularly unstable they simultaneously become ordinarily unstable. This last possibility was overlooked by Jeans who thought that secular instability must always come first. In the present book the importance of Cartan's results is emphasized and interpreted correctly.

As the author points out, the original papers are obscure in some places and inconsistent in others; and the present book provides a valuable and useful clarification of the subject. *S. Chandrasekhar* (Williams Bay, Wis.).

Elenevskaya, N. B. Expansion of a perturbation function in a Fourier series with respect to the inclination. II. Expansion of a perturbation function in the spatial circular restricted problem of three points in a Fourier series with respect to the inclination. Akad. Nauk SSSR. Byull. Inst. Teoret. Astr. 5, 271–302 (1953). (Russian)

Šaraf, Š. G. Expansion of certain functions of the coordinates of elliptic motion in series up to the 9th power of the eccentricity. Akad. Nauk SSSR. Byull. Inst. Teoret. Astr. 5, 303–314 (1953). (Russian)

de Jekhowsky, Benjamin. Nouvelle méthode pour la détermination des orbites paraboliques. C. R. Acad. Sci. Paris 237, 970–972 (1953).

Dubošin, G. N. On the differential equations of motion of the satellites of planets. Akad. Nauk SSSR. Astr. Žurnal 30, 315–331 (1953). (Russian)

The author remarks that in all the classical papers dealing with the motion of the satellites of a planet one and the same system of differential equations (1) is used without a clear specification under what simplifying assumptions these equations are obtained, and that as a consequence of this it is not known to what extent they are fit to represent the planetocentric motion sought. The paper aims to clarify this question.

Consider a system which consists of the Sun S and $n+1$ major planets $P_0, P_1, \cdots, P_n$. Assume that S and $P_1, \cdots, P_n$ can be considered as particles attracting each other according to the Newtonian law of attraction. The planet P_0 is considered as an invariable rigid body with given mass distribution. Further assume that this planet has k satellites $T_1, T_2, \cdots, T_k$, the motions of which with respect to P_0 is to be determined. It is also assumed that all satellites may be considered as particles, attracting each other and the particles $S, P_1, \cdots, P_n$ according to the Newtonian law. Also, each element of the planet P_0 is assumed to be submitted to the same law. The form of the planet P_0 being arbitrary, one of its principal axes of inertia is assumed to conserve its direction with respect to the space. Two rectangular coordinate systems are introduced, one with the origin at the center of inertia of P_0 and axes in the directions of the principal axes of inertia of P_0, the other with the origin at S and the axes parallel to the corresponding axes of the first system.

A system of $3(k+n+1)$ differential equations of the second order (4), (5), and (6) is established which determine simultaneously the planetocentric motion of the satellites and the heliocentric motion of the planets. If, in the systems (5) and (6), all terms containing the masses of the satellites of P_0 as factors are omitted, a system of $3(n+1)$ differential equations is obtained which determines the heliocentric motion of the planets. These equations contain also terms depending on the perturbations caused by the form of the planet P_0. Neglecting these terms the systems (5) and (6) finally reduce to the basic equations of planetary motions in the classical celestial mechanics. Considering the heliocentric motion of the planets as known, the system (4) gives the planetocentric motion of the satellites of P_0.

The classical equations of motion of the satellites (1) are obtained from those of the system (4) by omitting (i) the terms depending on the perturbations caused by the planets $P_1, P_2, \cdots, P_n$, and (ii) the terms depending indirectly on the perturbations caused by the form of the planet P_0. Thus the equations (1) take into account only the direct influence of the perturbations caused by the form of P_0, the perturbations of the Sun and the perturbations caused by the interactions of the satellites themselves on each other. Among the terms to be omitted in (4) in obtaining (1) some are undoubtedly very small and can be neglected without any

further questioning. However, the omission of some other terms cannot be considered as sound without a preliminary analysis.

Integrating the system (4) in power series of small parameters (such as the perturbative masses of the satellites of P_0, the coefficient of compression of the planet P_0, the reciprocal values of the average distances of the Sun and the perturbative major planet to the center of inertia of P_0), the classical equations (1) are not even capable of reproducing the terms up to the second order with respect to these parameters. Therefore the classical equations of motion of the satellites can be used for the determination of perturbations of the first order only.

The differential equations of motion of the satellites of Jupiter and of Saturn and the motion of the Moon as a satellite of the Earth are discussed in details.

E. Leimanis (Vancouver, B. C.).

Sitnikov, K. A. On the possibility of capture in the problem of three bodies. Mat. Sbornik N.S. **32(74)**, 693–705 (1953). (Russian)

Assume that the state of motion of a system of three bodies P_1, P_2, P_3, subjected to mutual Newtonian attraction is given at the initial instant $t=0$. We shall say that a capture takes place in this system, if (i) all mutual distances between the bodies become infinite as $t\to-\infty$, and (ii) the distance between any two of them (say, between P_1 and P_2) remains finite, while the distances between P_3 and the other two bodies increase indefinitely as $t\to+\infty$. Šmidt [Doklady Akad. Nauk SSSR (N.S.) **58**, 213–216 (1947)] and Sizova [ibid. **86**, 485–488 (1952); these Rev. **14**, 589] gave numerical examples to show the possibility of capture in the general problem of three bodies and in the hyperbolic case of the restricted problem of three bodies, respectively.

In the present paper a system is given which consists of three unit masses and depends on three numerical parameters c, n, R subject to the conditions $c>0$, $R>1000\,c^2$. It is shown that for arbitrary but sufficiently large values of n and any admissible values of c and R the phenomenon of a capture occurs. Furthermore, this capture proves to be stable in the sense that in any other system differing little from the given one capture takes place also.

Consider an inertial coordinate system Oxy in the plane of motion and let the positions and velocities of the three bodies be defined at $t=0$ as follows: $P_{10}(-c/n, 0)$, $\bar v_{10}(0, n)$; $P_{20}(c/n, 0)$, $\bar v_{20}(0, -n)$; $P_{30}(-2R, 0)$, $\bar v_{30}=\bar v_{31}+\bar v_{32}$, where $|\bar v_{31}|=(n^2-n/2c)^{1/2}$, $\tan\varphi=4c(n^2-n/2c)^{1/2}$ (φ is the acute angle which the vector $\bar v_{31}$ makes with the positive x-axis) and $\bar v_{32}$ $(0, 1/R^{1/2})$. The unit of length is chosen in such a way that the gravitational constant assumes the value unity.

The proof that such a system leads to a capture proceeds as follows. Subdivide the infinite time interval into three partial intervals: $t<-4R/n$, $|t|\leqq 4R/n$, $t>4R/n$. Consider, in addition to the system P_1, P_2, P_3, an auxiliary system P_1', P_2', P_3' which differs from the first one by the fact that P_3' does not interact with P_1' and P_2', i.e. P_3' moves inertially, while P_1' and P_2' move subject to their mutual attraction. Theorem 1 states that for sufficiently large n and any $|t|<4R/n$, the positions of the corresponding bodies in both systems differ by a distance less than t^2/R^2 and the modulus of their velocity difference is less than $2|t|/R^2$. By means of this theorem it is shown that for n sufficiently large and $t=4R/n$ the assumptions of Theorem 2 are satisfied. From this it follows that the distance between P_1 and P_3 remains less than $8R$, while the other two distances P_3P_1 and P_3P_2 become infinite as $t\to+\infty$. Further, it is shown that for $t=-4R/n$ and n sufficiently large the assumptions of the Theorem 3.3 of G. F. Hil'mi [The problem of n bodies in celestial mechanics and cosmogony, Izdat. Akad. Nauk SSSR, Moscow, 1951, p. 75 (unavailable for review)] are satisfied. From this theorem, the proof of which is reproduced in the paper, it follows that all the mutual distances become infinite as $t\to-\infty$. *E. Leimanis.*

Merman, G. A. New criteria of hyperbolic and hyperbolic-elliptic motion in the problem of three bodies. Akad. Nauk SSSR. Astr. Žurnal **30**, 332–339 (1953). (Russian)

This paper is a continuation of an earlier paper by the same author [Doklady Akad. Nauk SSSR (N.S.) **85**, 727–730 (1952); these Rev. **14**, 590]. New criteria of realizability of hyperbolic and hyperbolic-elliptic motion in the problem of three bodies are established. It is pointed out that they can be regarded as an improvement of the criteria obtained previously by the author (loc. cit.) and G. F. Hil'mi [The problem of n bodies in mechanics and cosmogony, Izdat. Akad. Nauk SSSR, Moscow, 1951] in the sense that the restrictions imposed in these papers on the radial velocities of the bodies and their initial configuration are removed. Instead it is required that the corresponding full velocities be sufficiently large. The new criteria are too lengthy to be reproduced here. They are applied to the example given by O. Yu. Šmidt [Doklady Akad. Nauk SSSR (N.S.) **58**, 213–216 (1947)] of illustrating the possibility of a capture in the problem of three bodies. *E. Leimanis.*

Radzievskiĭ, V. V. The spatial case of the restricted problem of three radiating and gravitating bodies. Akad. Nauk SSSR. Astr. Žurnal **30**, 265–273 (1953). (Russian)

The author investigates the spatial motion of an infinitely small particle in the photogravitational field of two finite bodies revolving in circular orbits with constant angular velocity about their common center of mass. It is shown that the surfaces of zero velocity of the particle may have seven double points. The two new libration points L_6 and L_7, called coplanar points, lie outside the plane of motion of the two finite bodies. A detailed investigation is made of the axes of libration, being the geometrical locus of points of libration for particles of various sizes, and of the passage of the Earth through the axes of libration of the system Sun-Jupiter-particle for the interval 1951–1955 [cf. also Radzievskiĭ, same Žurnal **27**, 250–256 (1950); **28**, 363–372 (1951); these Rev. **12**, 448]. *E. Leimanis.*

Lindblad, Bertil, and Langebartel, Ray G. On the dynamics of stellar systems. Stockholms Observatoriums Annaler **17**, no. 6, 61 pp. (2 plates) (1953).

The propagation of a wave like disturbance is considered in an oblate stellar system which is rotating with a constant angular velocity. An "adiabatic" theorem derived in a previous paper [same Annaler **16**, 1–34 (1950); these Rev. **12**, 754] is applied. According to this theorem the variation of density $\delta\rho$ and the variation of the velocity dispersion $\delta\alpha^2$ are connected by the formula $\delta\rho/\rho=\delta\alpha^2/\alpha^2$. Waves of the type $f(r)e^{i(\sigma t+s\theta)}$ are investigated, where θ denotes the azimuth in the equatorial plane. The stellar system is considered as a "quasi-spheroidal" system of radius $r=a$, in the sense that the system may be divided up schematically into a series of homogeneous spheroids with decreasing thickness for increasing density. The velocity dispersion in the equatorial plane is assumed to be a function of the

distance r from the axis of rotation and of the form $a^2=\alpha_0^2(1-r^2/a^2)$. The conditions of instability are discussed for the harmonics $s=1, 2, 3$ for various values of flattening. Along the lines of previous investigations the unstable mode of harmonic $s=2$ is regarded as "dissolving" wave in the development of spiral structure. *S. Chandrasekhar.*

Neyman, Jerzy, and Scott, Elizabeth L. Frequency of separation and of interlocking of clusters of galaxies. Proc. Nat. Acad. Sci. U. S. A. **39**, 737–743 (1953).

Consider a particular cluster Γ_1 of galaxies and denote by C_1 its center. Let ν_1 be the (random) number of galaxies belonging to Γ_1. These galaxies will be numbered in the order of their distance from C_1, beginning with the most distant galaxy. Let η_k denote the distance of the kth galaxy from the center. If a galaxy belonging to a cluster Γ_2 is located within the sphere of radius η_k (centered at C_1 of Γ_1) then Γ_2 is said to penetrate Γ_1 to the depth k. By the distance between two clusters of galaxies is meant the distance between their centers. And finally, associated with each cluster of galaxies there is a conditional probability density represented by a continuous function $f(\eta)$.

Let Γ^* be the cluster that is nearest to Γ_1, and let ν^* be the number of galaxies belonging to Γ^*. With these definitions a typical problem considered is the random variable $X(k)$ (defined for $1\leqq k\leqq\nu$) as the number of galaxies of the cluster Γ^* that penetrate Γ_1 to the depth k. The following result (without proof) is stated. The conditional probability generating function of $X(k)$, given that $\nu_1\geqq N_1\geqq 1$ and given that ν^* satisfies a certain hypothesis H, is given by

$$(1)\quad G_{X(k)}[t\,|\,(\nu_1\geqq N_1)H]=\int_0^1 dv\Phi(v,k,N_1)\int_0^1 du G_\nu(\tau\,|\,H),$$

where

$$(2)\quad \Phi(v,k,N_1)=\frac{1}{(k-1)!P\{\nu_1\geqq N_1\}}(1-v)^{k-1}\times\sum_{n=N_1}^{\infty}\frac{n!}{(n-k)!}v^{n-k}P\{\nu_1=n\},$$

$$(3)\quad \tau=1-(1-t)F[x(u),y(v)]$$

$$(4)\quad F(\xi,\eta)=\iiint_{r\leqq 1} f\{[(x-\xi)^2+y^2+z^2]^{1/2}\}dxdydz$$

with $x(u)$ and $y(v)$ defined for $0\leqq u, v<1$ by

$$(5)\quad x(u)=\left\{\frac{-3\log(1-u)}{4\pi\lambda P(H)}\right\}^{1/3}.$$

In the foregoing expressions, $G_\nu(t)$ is the probability-generating function of an integer-valued random variable ν (so that $G_\nu(t)$ is the expectation value of t^ν), $P(H)$ is the probability that the number ν of galaxies in a cluster will satisfy the hypothesis H, and λ is the expected number of cluster centers per unit volume.

Similar results relating to the number of clusters which have some galaxies penetrating a given cluster and the number of galaxies that are "imbedded" in a given cluster are also stated. *S. Chandrasekhar* (Williams Bay, Wis.).

Pack, D. C. The motion of a gas cloud expanding into a vacuum. Monthly Not. Roy. Astr. Soc. **113**, 43–51 (1953).

The motion of a semi-infinite one-dimensional gas cloud adjoining a vacuum has been considered under various conditions by Burgers [Nederl. Akad. Wetensch., Proc. **49**, 589–607 (1946)], McVittie [Monthly Not. Roy. Astr. Soc. **110**, 224–237 (1950); these Rev. **12**, 642] and Copson [ibid. **110**, 238–246 (1950); these Rev. **12**, 643]. These problems are reconsidered in this paper by the method of characteristics. The advantage of this latter method consists in the fact that one can readily visualize the nature of the solutions from an examination of the characteristic diagram. It is further shown that unless certain conditions are satisfied by the initial state in the cloud the subsequent flow cannot remain indefinitely continuous: shock waves must appear in the gas. An example of this latter breakdown of continuity is given. *S. Chandrasekhar.*

Busbridge, Ida W. Coherent and noncoherent scattering in the theory of line formation. Monthly Not. Roy. Astr. Soc. **113**, 52–66 (1953).

The basic problem which is considered in this paper is the law of diffuse reflection by a semi-infinite atmosphere scattering isotropically with an albedo λ, depending on the frequency ν such that the probability $\rho(\nu,\nu')$ that radiation of frequency ν is scattered as radiation of frequency ν' is of the form $\sigma(\nu)\sigma(\nu')$ where $\sigma(\nu)$ is a function of ν only. It is evident that in this case the law of diffuse reflection can be expressed in terms of H-functions in the usual form [cf. S. Chandrasekhar, Radiative transfer, Oxford, 1950; these Rev. **13**, 136]. The importance of the considerations of this paper arise from the fact that it can be used to solve the problem of line formation in which we have a special case of non-coherent scattering. *S. Chandrasekhar.*

Bleksley, A. E. H. A new approach to cosmology. IV. South African J. Sci. **50**, 61–64 (1953).

For parts I–III see same J. **46**, 343–346 (1950); **47**, 13–15 (1950); **48**, 20–24 (1951); these Rev. **12**, 449; **13**, 290.

RELATIVITY

***Einstein, Albert. Supplement to Appendix II of "The meaning of relativity, 4th ed."** Princeton University Press, Princeton, N. J., 1953. 8 pp.

In this publication Einstein reviews the concept of the "strength" of a system of equations and recalculates the effect the freedom of choice of the coordinate system has on the strength of a set of covariant field equations thus correcting an error in Appendix II of the 4th edition of "The meaning of relativity" [Princeton, 1953; these Rev. **14**, 805]. He also gives another derivation of his generalised field equations which uses a variational principle and an argument which differs from that given in the publication referred to above. The main difference is that the affine connection is written as a sum of two parts:

$$\Gamma^*_{kl}=\Gamma^{**}_{kl}+\delta_k{}^s\lambda_l$$

where λ_l is a vector. The Γ^{**}_{kl} and λ_l are varied independently and λ_l is chosen so that $\Gamma_k{}^*=\frac{1}{2}(\Gamma^{**}_{sk}-\Gamma^{**}_{ks})$. The resulting field equations are identical with those obtained earlier with the same Lagrangian but using variations subject to an auxilliary condition.

A. H. Taub (Zurich).

***Lichnerowicz, André. Etude mathématique des théories relativistes de la gravitation et de l'électromagnétisme. I. Relativité générale classique.** (Cours professé au Collège de France pendant l'année scolaire 1952–1953.) Paris, 1953. 149 pp. (mimeographed)

In the introduction to these lectures the author states his view that general relativity, now nearly forty years old, has passed from theoretical into mathematical physics and should be treated with mathematical rigor like the theory of potential. In this spirit he presents relativity axiomatically, with much more attention to mathematical precision than has been customary and much less attention to its physical aspects. Space-time is covered with overlapping admissible coordinate systems, with transformations of class C^2 in the overlaps, and the gravitational potentials $g_{\alpha\beta}$ are postulated to be of class C^1. This postulate supplies the conditions of junction (raccordement) across a 3-space on which the second derivatives of $g_{\alpha\beta}$ are discontinuous (passage from matter to vacuum), thus obviating recourse to limiting processes [cf. O'Brien and Synge, Communications Dublin Inst. Advanced Studies. Ser. A. no. 9 (1952); these Rev. 14, 913]. Part I deals with Einstein's equations and the energy-momentum tensor, Cauchy's problem for the equations of gravitation and electromagnetism, and junction conditions; the basic postulates are shown to imply that Gaussian coordinates (for which $g_{10}=g_{20}=g_{30}=0$, $g_{00}=\pm 1$) are always admissible, i.e. the first order partial derivatives of $g_{\alpha\beta}$ are continuous. Part II gives a full treatment of relativistic hydrodynamics, including the Eisenhart stationary principle for a perfect fluid, relative integral invariant, vorticity, permanent motions, Bernoulli's theorem, a tentative theory of viscous fluids, and the theory of perfect charged fluids. Part III is devoted to a global theory of stationary space-time, including calculation of the Ricci tensor of a space V_{n+1} admitting a 1-parameter group of isometries, Laplace's equation in Riemannian space with positive-definite metric, and stationary space-times regular everywhere. *J. L. Synge* (Dublin).

Papapetrou, Achilles. Die Relativitätstheorie und ihre neuere Entwicklung. Wissensch. Ann. 2, 593–609 (1953). Expository paper.

Papapetrou, A. Eine rotationssymmetrische Lösung in der allgemeinen Relativitätstheorie. Ann. Physik (6) 12, 309–315 (1953).

Es handelt sich um das stationäre Gravitationsfeld eines mit konstanter Winkelgeschwindigkeit um die Symmetrieachse rotierenden Körpers. Eine spezielle Lösung, die das Feld eines unendlich langen Zylinders beschreibt, der gleichförmig mit Masse belegt ist, hat Lewis angegeben [Proc. Roy. Soc. London Ser. A. 136, 176–192 (1932)]. Verf. gibt eine neue exakte Lösung. Es zeigt sich, dass diese Lösung das Feld eines rotierenden Körpers darstellt, der zwar endliche Ausdehnung hat, aber verschwindende Gesamtmasse besitzt. *J. Haantjes* (Leiden).

Kalitzin, Nikola St. Elektromagnetismus und Gravitation. C. R. Acad. Bulgare Sci. 4 (1951), no. 2–3, 13–16 (1953). (Russian summary)

Essentially a briefer version of what is given in Kalitzin, Izvestiya Bulgar. Akad. Nauk. Otd. Fiz.-Mat. Tehn. Nauk. Ser. Fiz. 2, 49–62 (1952); these Rev. 15, 169. *N. Rosen.*

Bonnor, W. B. Nonsymmetric unified field theories. Physical Review (2) 92, 1067–1068 (1953).

Takasu, Tsurusaburo. The general relativity as a three-dimensional non-holonomic Laguerre geometry of the second kind, its gravitation theory and its quantum mechanics. Yokohama Math. J. 1, 89–104 (1953).

In the first part of this paper the formulae of general relativity are brought into correspondence with the "non-holonomic" Laguerre geometry of the second kind developed in a former publication [same J. 1, 1–28, 29–38, 39–74, 75–77, 79–82, 83–87 (1953); these Rev. 15, 350]. In the second part general relativistic quantum mechanics is dealt with from this point of view. As a result physics is looked upon as "Kugelgeometrie" and physical phenomena are considered as phenomena that "take place in our Cartesian space, in which several branches of 'Kugelgeometrie' are realized, each branch describing a physical phenomenon". As to classification, "physics is classified into branches corresponding to the branches of the geometry it is based upon". The author announces that these viewpoints are enabling him to describe enormously vast new ranges of physical phenomena. *J. A. Schouten* (Epe).

Mátrai, T. A relativistic treatment of rigid motion. Nature 172, 858–859 (1953).

Stueckelberg, E. C. G., et Wanders, G. Thermodynamique en relativité générale. Helvetica Phys. Acta 26, 307–316 (1953).

The basic equation of this paper is

$$(*) \qquad \epsilon v^\alpha D_\beta \Theta_\alpha{}^\beta + \sum_A \mu_A D_\beta n_A{}^\beta + T(D_\beta s^\beta - i) = 0.$$

Here v^α is 4-velocity ($v_\alpha v^\alpha = -\epsilon$; $\epsilon^2 = 1$), D_β denotes covariant differentiation, $\Theta_\alpha{}^\beta$ is momentum-energy current, $A = 1, 2, \cdots$ is an index denoting a particular constituent substance, μ_A its chemical potential, $n_A{}^\beta$ its flux of quantity, T temperature, s^β entropy vector, and i (≥ 0) is called the irreversibility. The plan is to assume a form for $\Theta_\alpha{}^\beta$ and use (*) as a formal identity; v^α, μ_A, T are taken as variables of state. For a perfect fluid the authors take $\Theta^{\alpha\beta} = mv^\alpha v^\beta + \epsilon p g^{\alpha\beta}$ with m and p functions of T, μ_A, and the first term of (*) is then (**) $-v^\alpha D_\alpha \phi - m D_\alpha v^\alpha$ where $\phi = m - p$. They introduce variables σ, ν_A conjugate to T, μ_A by means of $T = \partial\phi/\partial\sigma$, $\mu_A = \partial\phi/\partial\nu_A$ and are led to write $m = T\sigma + \sum_A \mu_A \nu_A$, thus giving to (**) the form

$$-T D_\alpha(\sigma v^\alpha) - \sum_A \mu_A D_\alpha(\nu_A v^\alpha).$$

Substitution of this in (*) leads them to $n_A{}^\alpha = \nu_A v^\alpha$, $s^\alpha = \sigma v^\alpha$, $i = 0$. This must serve as an indication of the nature of the paper, which continues with a discussion of viscosity, thermal conduction, diffusion and chemical reactions, suitable expressions for $\Theta_\alpha{}^\beta$ being substituted in (*).

J. L. Synge (Dublin).

See the note on p. 1140

***Leaf, Boris. The continuum in special relativity. II.** Physical Review (2) 90, 1090–1101 (1953).

This paper generalises the theory of a previous paper [Physical Review (2) 84, 345–350 (1951); these Rev. 13, 500] and deals with the equation of continuity of a continuum in special relativity, equations of motion, entropy tensor and second law of thermodynamics, stress and reversibility, and transport processes in a one-component medium. The generalisation consists in replacing the usual conservation equations $\partial_\sigma \psi_{\sigma\tau} = 0$ ($\psi_{\sigma\tau}$ = energy tensor in primary frame of reference) by (*) $\partial_\sigma{}^* \psi_{\sigma\tau} = 0$ where $\partial_\sigma{}^*$ means partial differentiation with respect to coordinates in a secondary frame

in uniform motion relative to the primary frame. [The author states that the secondary frame is arbitrary (except for a restriction imposed near the end of the paper). Thus (*) contains the coefficients of an arbitrary Lorentz transformation, and if we take that transformation infinitesimal, it is easy to see that (*) implies that ψ_{rr} is of the form $\psi_{rr}=\partial_r\partial_r\chi$ where χ is some scalar satisfying $\partial_r\partial_r\chi=\text{const.}$ Such a restriction on the form of the energy tensor would make the theory quite uninteresting and cannot have been intended by the author, who appears to have overlooked these facts.]

J. L. Synge (Dublin).

Jacoby, G. Die ontologischen Hintergründe der speziellen Relativitätstheorie. Wissensch. Z. Univ. Greifswald. Math.-Nat. Reihe **2**, 237–250 (1953).

MECHANICS

***Garnier, René. Cours de cinématique. Tome I. Cinématique du point et du solide, composition des mouvements.** 3ième éd. Gauthier-Villars, Paris, 1954. ix+244 pp. 4000 francs.

The 2d edition of 1949 was reviewed in these Rev. **13**, 381. This edition differs only in minor simplifications, additions, and new figures.

Dobrovol'skiĭ, V. V. The construction of the relative positions of the links of spatial seven-bar linkages by the method of spherical representation. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **12**, no. 47, 52–62 (1952). (Russian)

The general three-dimensional closed-chain linkage mechanisms having only one degree of freedom are examined. These are the seven-bar with only hinged joints, the six-bar with one hinge replaced by a cylindrical joint (this permits both rotation and sliding), the five-bar with two cylindrical joints, and the four-bar with three cylindrical joints. Configurations of a mechanism may be described by vectors which represent the constant length and variable direction of the common perpendiculars between the adjacent joints, and the vectors for the variable length and direction of the offsets (distance between the feet of the perpendiculars on a joint).

The method described is the use of a spherical representation for the mutual angles which, in turn, is represented by its stereographic projection on a plane. The angle between the axes of rotation at the ends of a bar is called the twist. The representation breaks down into one spherical polygon for the twists and another for the offsets. Several relations between them are derived. *M. Goldberg.*

Müller, Hans Robert. Verallgemeinerung der Bresseschen Kreise für höhere Beschleunigungen. Arch. Math. **4**, 337–342 (1953).

Verfasser beweist unter Anwendung komplexer Zahlen die zwei folgenden Sätze, welche Erweiterungen bekannter Theoreme der ebenen Kinematik sind. 1) Alle Punkte der Gangebene, deren Beschleunigungsvektoren $(m-1)$ter und $(n-1)$ter Ordnung einen festen Winkel λ einschliessen, liegen auf einem Kreis durch die Beschleunigungspole P_m, P_n der Ordnung $(m-1)$ bzw. $(n-1)$. 2) Alle Punkte der Gangebene, deren $(m-1)$te und $(n-1)$te Beschleunigungsvektoren ein festes Längenverhältnis besitzen, erfüllen gleichfalls einen Kreis (der auf dem in 1) genannten senkrecht steht). Es zeigt sich dass die behaupteten Sätze auf Eigenschaften ähnlicher Felder zurückgehen.

O. Bottema (Delft.).

Cattaneo, Carlo. Sulla statica dei fili perfettamente flessibili resistenti a torsione. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **16(85)**, 27–46 (1952).

The equations of equilibrium of an inextensible and perfectly flexible cable under tension and torsion are given in the intrinsic form (5): $(Tt+\Gamma b/\rho)'+f=0$. Here T is the tension, Γ the torsion torque, t and b the unit tangent and binormal, ρ the curvature and f the linear force density. The prime denotes derivative with respect to the arc. This equation is rewritten in Cartesian, and then in arbitrary, point coordinates using tensor notation and covariant derivatives. For a cable on a surface these equations are adapted by the usual fitting of the coordinate system (perhaps with excessive formalism and detail). The simple result [obtainable from (5) and Meusnier's theorem without a detour] is that the normal (in the surface) component of the applied force density is $c_\rho T+\Gamma(c_\sigma c_n+c_n')$. [The author's alternate sign in front of Γ can be eliminated by an intrinsic adjustment of senses.] It appears that a geodesic is a equilibrium figure of an unsolicited cable on a smooth surface only if it is simultaneously a line of constant normal curvature. A free cord solicited only at its ends will be helical. A cord under parallel distributed forces and a small torsion torque Γ is treated as a perturbation of the case $\Gamma=0$. The tension remains the same, and the angle of the cord with the plane of original equilibrium is $\Gamma T/\rho$ where ρ is the original curvature. *A. W. Wundheiler* (Chicago, Ill.).

Müller, Henning. Zur nichtlinearen Theorie der schwingenden Saite. Ann. Physik (6) **12**, 398–400 (1953).

Equations of motion are derived for a vibrating, but not necessarily straight string. The author discusses the approximation to these equations due to Kirchhoff in which one nonlinear term is added to the linear wave equation arising in the theory of small amplitude vibrations of a straight stretched string. *E. Pinney* (Berkeley, Calif.).

Signorini, Antonio. Sopra un particolare tipo di moti rigidi sferici. Rivista Mat. Univ. Parma **3**, 307–316 (1952).

The author gives an elegant geometrical approach to various matters connected with the frictionless motion of a rigid body about a fixed point O distinct from the center of gravity G. The geometrical considerations are concerned with antipolarity with respect to the so-called "central ellipse" of the arbitrary plane P, which is but a slight modification of the ellipse in which the ellipsoid of inertia, about the orthogonal projection of G on the plane P, cuts the plane P. Analogous questions were treated by Manacorda [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 711–714 (1949); these Rev. **11**, 549].

D. C. Lewis, Jr. (Baltimore, Md.).

***Consiglio, Alfonso. Sopra la dinamica di un sistema di punti vincolati con vincoli anolonomi.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 486–500. Casa Editrice Perrella, Roma, 1953.

A rapid and simple development of results on ignorable coordinates, Hamilton's principle, and generalized potential functions for conservative non-holonomic systems.

D. C. Lewis, Jr. (Baltimore, Md.).

Castoldi, Luigi. Permanenza della funzione hamiltoniana di un sistema lagrangiano generale attraverso trasformazioni di Routh. Atti Accad. Ligure **8** (1951), 356–362 (1952).

It is shown that the Hamiltonian corresponding to the Lagrangian obtained from the transformation of Routh is identical with the Hamiltonian corresponding to the Lagrangian of the original system. *D. C. Lewis, Jr.*

Hydrodynamics, Aerodynamics, Acoustics

Aržanyh, I. S. Investigations in the mechanics of a continuous medium. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **9**, 60–101 (1952). (Russian)

The author generalizes results obtained previously [Doklady Akad. Nauk SSSR (N.S.) **73**, 41–44; **74**, 21–24 (1950); these Rev. **12**, 651, 367] concerning the solution of boundary-value problems in fluid dynamics and elasticity. He considers a variety of problems involving the flow of ideal gases, ideal liquids, and viscous liquids as well as static and dynamic problems in elasticity, deriving integral equations whose solution yields a solution to the problem considered.

His "vortex interpretation of the theory of functions of a complex variable" [ibid. **73**, 667–670 (1950); these Rev. **12**, 646] is discussed in some detail. In essence, this is a superposition principle for ideal liquids similar to that for ideal gases obtained by Prim [Quart. Appl. Math. **7**, 445–450 (1950); these Rev. **11**, 271].

If the motion of a homogeneous rigid body T is resolved into a translational velocity $\vec{U}$ of one of its points and an angular velocity $\vec{\omega}$ about an axis through this point, the velocity potential φ of the irrotational flow of an ideal liquid surrounding T, which is produced by the motion of T, is given by $\varphi=(\vec{f}, \vec{U})+(\vec{g}, \vec{\omega})$, where $\vec{f}$ and $\vec{g}$ are vectors determined by the geometry of T. The author relates the gravitational force per unit mass due to T at a point exterior to T and the moment of this force in the point moving with velocity $\vec{U}$ to $\vec{f}$ and $\vec{g}$, respectively.

Finally, the author obtains a general solution to the dynamical equations of linear elasticity in terms of solutions to the wave equation. *J. L. Ericksen.*

Truesdell, Clifordo. La velocità massima nel moto di Gromeka-Beltrami. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **13**, 378–379 (1952).

Dans le mouvement d'un milieu continu quelconque soit w la grandeur du vecteur tourbillon et d_{ij} le tenseur des vitesses de déformation; l'auteur considère le nombre W défini par $W=w(2d^{ij}d_{ij})^{-1/2}$; ce nombre W sans dimensions que l'auteur appelle "numero di vorticità" mesure le rapport de la grandeur de la rotation locale à la "grandeur" de la déformation. Dans un milieu continu effectuant un mouvement de Gromeka-Beltrami, c'est-à-dire un mouvement pour lequel les lignes de tourbillons coïncident avec les lignes de courant, le Laplacien du carré de la grandeur de la vitesse s'exprime très simplement en fonction de W et des grandeurs cinématiques élémentaires (vitesse, tourbillon, vitesse de dilatation cubique θ). En utilisant ce résultat et les propriétés des fonctions sur-harmoniques et sous-harmoniques, l'auteur obtient le théorème suivant: le milieu continu effectuant un mouvement de Gromeka-Beltrami, si dans une région $W\leq 1$ et si θ ne décroît pas le long des lignes de courant dans le sens du mouvement, la grandeur de la vitesse ne peut passer par un maximum en un point intérieur de la région. Si dans la proposition précédente on renverse le sens des inégalités ou si on les remplace par des égalités, on obtient deux autres propositions analogues à celle énoncée. Par ce théorème, des résultats de Bouligand [Verh. 2. Internat. Kongresses Tech. Mech., Zürich, 1926, Füssli, Zürich, 1927, pp. 460–461] et de Hamel [Monatsh. Math. Phys. **43**, 345–363 (1936)] se trouvent être généralisés. Pour un mouvement irrotationnel et pour un mouvement isochorique ($\theta=0$) on obtient des corollaires particulièrement simples. L'auteur donne de très nombreuses autres propriétés du nombre W dans un mémoire qui a paru ailleurs [J. Rational Mech. Anal. **2**, 173–217 (1953); ces Rev. **14**, 1026]. *R. Berker* (Istanbul).

Woods, L. C. The two-dimensional subsonic flow of an inviscid fluid about an aerofoil of arbitrary shape. I–IV. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2811** (1950), 58 pp. (1953).

The paper describes and applies exact methods of calculating the incompressible flow about thick aerofoils of general shape in a free stream, and about symmetrical aerofoils between channel walls. One of these methods is extended to an approximate treatment of subsonic compressible flow by making use of von Kármán's transformation. *Author's summary.*

Söhngen, Heinz. Luftkräfte an einem schwingenden Schaufelkranz kleiner Teilung. Z. Angew. Math. Physik **4**, 267–297 (1953).

The flow is approximated as two-dimensional, inviscid, and incompressible. The cascade of blades is assumed to be made up of flat plates closely spaced, so that the basic, steady flow consists of a parallel flow that turns through an angle, not necessarily small, at the leading edges. This is accomplished by a row of vortices at the leading edges, and the flat blades lie parallel to the parallel stream downstream of this row. Under oscillatory conditions the leading-edge vortices move through small distances and vary slightly in strength; there also appear weak unsteady vortex distributions along the plates, which can be taken to lie along the mean positions of the plates. The boundary conditions to be met are imposed at the oscillating plates, and are linearized in the usual way, assuming small amplitudes.

The author has noticed that if each elementary vortex of variable strength is imagined to have a trailing wake of shed vortices behind it, the simple Kutta-Joukowski formula gives the force on it, as in steady flow. The blade, then, consists of these "bound" vortices and their wakes ("free" vortices); the velocity field is due to both categories. This is equivalent to Küssner's use of "effective" vortices [Luftfahrtforschung **13**, 410–424 (1936)] or the use of the acceleration potential. [See also Munk, J. Aeronaut. Sci. **5**, 454 (1938).] It differs from the more common concept of a bound-vortex sheet followed by a trail of wake vortices shed at the trailing edge. Throughout, summation over the infinitely many blades is replaced, approximately, by integration; the approximation is that of close spacing and is called "smearing" the blades and their wakes.

For harmonic oscillations the resulting integral equation can be solved. Results are worked out in detail for bending oscillations (1) normal and (2) parallel to the plates and (3) torsional oscillations about the leading edges. The blades are not assumed to vibrate in phase, but with a periodic pattern along the blade row. For these cases lift and moment are calculated. The results are applied to an investigation

of blade flutter in pure bending of an unstaggered cascade. The critical flutter speed is found and it is concluded that this type of flutter of a loaded blade wheel may occur in practice. *W. R. Sears* (Ithaca, N. Y.).

***Kravtchenko, Julien. Sur le problème indéterminé des sillages.** Comptes Rendus du Congrès des Sociétés Savantes de Paris et des Départements tenu à Grenoble en 1952, Section des Sciences, pp. 53–56. Gauthier-Villars, Paris, 1952.

Cf. Ann. Inst. Fourier Grenoble **3**, 287–299 (1952); these Rev. **14**, 325.

Ivanova, L. S. The added mass of a fluid filling an open rectangular basin. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 491–495 (1953). (Russian)

Dans le plan Oxy, considérons le domaine: $-l_1/2 \leqq x \leqq l_1/2$; $0 \leqq y \leqq l_2$; ce sera la section longitudinale d'un vase, ouvert, parallélépipédique, rempli d'un liquide parfait, non pesant, au repos pour $t \leqq 0$. La surface $y=l_2$ est libre. A l'instant initial $t=0$, le vase prend un mouvement de translation de vitesse $\mathbf{v}_0$, parallèle à Ox. L'auteur détermine le champ initial des vitesses du liquide, en déduit l'impulsion subie par le vase et discute les résultats. *J. Kravtchenko.*

Myasnikov, P. V. On the pressure of a free plane jet on an obstacle. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk **1950**, no. 6, 3–20 (1950). (Russian)

L'auteur reprend un schéma voisin de celui des sillages classiques de Helmholtz-Kirchhoff et étudie les écoulements plans, permanents, à potentiel d'un liquide, le domaine du mouvement étant limité par des lignes libres et un obstacle rectiligne. L'auteur complète les formules bien connues en explicitant les valeurs de la pression et des composantes de la vitesse en chaque point de l'écoulement (en fonction des paramètres réels auxiliaires) et où les graphiques de distribution des pressions le long du profil immergé indiquent les formules approchées suffisantes pour les applications. La partie expérimentale du mémoire paraît plus originale. L'auteur donne des conditions de réalisation au laboratoire de ses schémas et compare ses prévisions théoriques avec les résultats de ses essais. L'accord paraît satisfaisant.

J. Kravtchenko (Grenoble).

Woronetz, Konstantin. L'influence des forces extérieures sur l'écoulement par les orifices. Acad. Serbe Sci. Publ. Inst. Math. **5**, 183–194 (1953).

The author gives a general perturbation scheme for considering the effect of external forces in the Helmholtz-Kirchoff free-boundary theory. He treats briefly the first order effect of gravity on a (plane) jet issuing from an orifice. *J. B. Serrin* (Cambridge, Mass.).

***Schönfeld, J. C. Propagation of tides and similar waves.** Thesis, Technische Hogeschool te Delft, 1951. Staatsdrukkerijen Uitgeverijbedrijf, 's-Gravenhage, 1951. 232 pp. (12 plates). (Dutch and French summaries)

The author is concerned with the propagation of waves, particularly tidal waves, in river estuaries and canal systems. Such motion is essentially one-dimensional in space, and whether or not resistance effects are included, satisfies equations which are quasi-linear, i.e. of the form

$$A_{11}\frac{\partial z_1}{\partial x_1}+A_{12}\frac{\partial z_2}{\partial x_1}+B_{11}\frac{\partial z_1}{\partial x_2}+B_{12}\frac{\partial z_2}{\partial x_2}+C_1=0$$

$$A_{21}\frac{\partial z_1}{\partial x_1}+A_{22}\frac{\partial z_2}{\partial x_1}+B_{21}\frac{\partial z_1}{\partial x_2}+B_{22}\frac{\partial z_2}{\partial x_2}+C_2=0$$

where A, B and C are functions of x_1, x_2, z_1, and z_2. These equations can be transformed by a characteristic transformation, if certain conditions hold which are true in the case of wave propagation, to the simpler form

$$\partial f/\partial x_1+c\partial f/\partial x_2+a=0,$$

where f is a function of x_1, x_2, z_1, and z_2. The theory of these equations is described.

Applying this theory to the equations of wave propagation, some results about the nature of the solutions are derived, with and without the resistance effect. The treatment of a canal system as a network with equivalent impedances is also discussed. Three methods of solution of the equations are described in detail, and applied to a laboratory model of a canal with uniform cross-section. The accuracy of the solutions obtained by the various methods is compared, and recommendations made on the use of the various methods. The methods are: (i) a method of integrating along the characteristics, to be generally recommended; (ii) the method of solving the equations for various components by successive approximation, or "linearisation", i.e., first to neglect the non-linear resistance term and then include its effect by using the first approximation therein, and to repeat the procedure as necessary; and (iii) a step-by-step direct integration process using either Taylor's series or differences.

The treatment of an actual estuary with an irregular shaped channel is considered; the channel is here represented by a number of sections, in each of which the channel is considered to be uniform. The modifications arising when a bore exists are described, and the effect of the errors arising from the use of the method of characteristics, the neglect of non-linear terms, and the representation of a channel by uniform sections are estimated. *D. C. Gilles* (London).

Kampé de Fériet, J., and Kotik, J. Surface waves of finite energy. J. Rational Mech. Anal. **2**, 577–585 (1953).

This paper supplies the proof for a result announced earlier [C. R. Acad. Sci. Paris **235**, 230–232 (1952); these Rev. **13**, 997]. *J. V. Wehausen* (Providence, R. I.).

Thorne, R. C. Multipole expansions in the theory of surface waves. Proc. Cambridge Philos. Soc. **49**, 707–716 (1953).

The author derives the velocity potentials for irrotational motion of an inviscid fluid with the linearized free-surface condition and with singularities of the following kinds: in two dimensions:

$$\log r\cos\sigma t,\quad r^{-n}\cos n\theta\cos\sigma t,\quad r^{-n}\sin n\theta\cos\sigma t;$$

in three dimensions:

$$r^{-n-1}P_n^m(\cos\theta)\begin{Bmatrix}\cos m\alpha\\ \sin m\alpha\end{Bmatrix}\cos\sigma t,$$

where (r, θ) and (r, θ, α) are polar coordinates in two and three dimensions respectively. Both infinitely deep water and water of finite depth are treated. In each case the potential function is expressed both as the principal value of an integral and as a series expansion in r^2. The potential is chosen in each case so as to represent a diverging wave. For the cases $m=n=0$, the results are known.

J. V. Wehausen (Providence, R. I.).

Haskind, M. D. Oscillations of a floating contour on the surface of a heavy liquid. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 165–178 (1953). (Russian)

The author treats the oscillatory motion of a long cylindrical body (width $2a$ at the waterline) floating freely

in an infinitely deep inviscid fluid; the problem is linearized. The mathematical problem is to find a harmonic function $\Phi(x, y, t)=\varphi(x, y)e^{i\sigma t}$ such that (1) $\varphi_y-\nu\varphi=0$ ($\nu=\sigma^2/g$) for $y=0$, $|x|>a$, (2) $\varphi_n=v_n$ on the contour of the cylinder, and (3) φ has the asymptotic values $(ig\sigma^{-1}r_0+B_+)e^{-i\nu(x+iy)}$ as $x\to+\infty$ and $ig\sigma^{-1}r_0e^{-i\nu(x+iy)}+B_-e^{+i\nu(x-iy)}$ as $x\to-\infty$. For a wide class of body profiles formulas for the force and moment on the body are derived. *J. V. Wehausen.*

Haskind, M. D. The diffraction of waves about a moving cylindrical vessel. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 431–442 (1953). (Russian)

An infinitely long cylindrical vessel is partly immersed in a heavy fluid, so that the generators are parallel to the (undisturbed) free surface; the width at the free surface is $2a$. The cylinder is assumed to move with constant velocity u in the direction (positive x-axis) of its generators; sinusoidal waves are assumed to propagate from a distance, making an angle ϵ with the x-axis. The velocity u enters into the problem only in a mathematically trivial fashion: since the axes are taken fixed in the vessel, the frequency σ_0 of oncoming waves is modified to $\sigma=\sigma_0-ku\cos\epsilon$. The velocity potential $\Phi(x, y, z, t)$ is taken in the form

$$\psi(y, z)\exp i(\sigma t-kx\cos\epsilon).$$

Then the function ψ must satisfy the usual conditions:

$$\psi_{yy}+\psi_{zz}-k_1^2\psi=0 \quad (k_1=k\cos\epsilon),$$
$$\psi_z-k\psi=0 \quad \text{for } z=0 \text{ and } |y|>a,$$
$$\psi_n=0 \quad \text{on } L, \text{ the contour of the cylinder,}$$

and also the conditions:

$$\psi=ig\sigma_0^{-1}r_0\exp(kz-iky\sin\epsilon)+C_\pm\exp(kz\mp ik_2y) \quad \text{as} \quad y\to\pm\infty,$$

where $k_2=k|\sin\epsilon|$. The function ψ is expressed in several ways in terms of an auxiliary function f related to ψ by $f_z=\psi_z-k\psi$. For two special cases the author finds the function f and derives the force and moment on the cylinder: (1) a long vertical flat plate submerged to depth T; (2) a long horizontal flat plate of width $2a$. In each case elliptic coordinates are introduced and the solutions are in terms of Mathieu functions. *J. V. Wehausen.*

Kostyukov, A. A. On the formation of waves in the motion of a ship. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 275–284 (1953). (Russian)

For the case of a moving body submerged in a heavy fluid, Kočin [e.g., Trudy konferencii po teorii volnovogo soprotivleniya, Izd. Central. Aèro-Gidro. Inst., 1937, pp. 65–134, esp. pp. 95–101, 126–133; cf. these Rev. 13, 997] showed that the velocity potential can be written

$$\varphi(x, y, z)=\frac{1}{4\pi}\int\int_S\left[\frac{1}{r}-\frac{1}{r'}+K(x-\xi, y-\eta, z+\zeta)\right]q(\xi, \eta, \zeta)dS$$

where q is determined by an integral equation:

$$q(x, y, z)=\int\int_S L_\nu(x, y, z, \xi, \eta, \zeta)q(\xi, \eta, \zeta)dS-2v\cos(n, x).$$

Here r and r' are the distances between (x, y, z) and (ξ, η, ζ) and $(\xi, \eta, -\zeta)$ respectively, S is the surface of the body, v its (constant) velocity, and $\nu=g/v^2$ is a parameter upon which the kernels depend. Kočin [loc. cit., pp. 126–133] showed that a solution exists for the integral equation if ν is sufficiently large. Using an iterative scheme somewhat different from Kočin's, the author shows that the integral equation has a solution for both large and small ν for both submerged and nonsubmerged (S=wetted surface) bodies. q is computed and shown graphically for a ship of infinite draft and parabolic waterline. He also derives asymptotic expressions for the shape of free surface for large and small ν and derives Michell's integral for the wave resistance from an expression in terms of q given by Kočin [loc. cit., p. 110]. *J. V. Wehausen* (Providence, R. I.).

Sretenskiĭ, L. N. Waves of finite amplitude arising from a periodically distributed pressure. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1953**, 505–511 (1953). (Russian)

The author treats the problem of the title in the following fashion, restricting himself to an infinitely deep fluid and two-dimensional motion. Consider a velocity potential $w=-cz+\gamma k^{-1}e^{-ikz}$, $z=x+iy$, γ and k real, $k>0$. Let $w'=-u+iv$, $\zeta=-ikz$, $\tau=qt$ (q real). Then an equation for the paths of individual particles is given by $d\bar{\zeta}/d\tau=i\nu-\epsilon e^{\zeta}$, where $\nu=kc/q$, $\epsilon=k\gamma/q$. The author assumes ϵ small, that $\nu=1+\epsilon\nu_1+\epsilon^2\nu_2+\cdots$, and looks for ζ in the form $\zeta=\zeta_0+\epsilon\zeta_1+\epsilon^2\zeta_2+\cdots$. Substitution in the differential equation for $\bar{\zeta}$ gives a sequence of equations for the ζ_i which may be solved successively, the ν_i being selected so that the solutions after ζ_1 are periodic functions of τ (thus, $\nu_1=\nu_2=0$). Carried to ζ_3, this gives $\nu=1+\epsilon^3\exp[i(\alpha-\bar{\alpha})]$ and

$$\zeta=-i(\tau+\bar{\alpha})+\epsilon[ie^{i(\tau+\alpha)}]+\epsilon^3[\tfrac{1}{2}ie^{2i(\alpha-\bar{\alpha})}e^{-i(\tau+\alpha)}]+\cdots,$$

where $\alpha=a+ib$ is a constant. From this one obtains formulas for x and y, different trajectories corresponding to different choices of b. Substitution in Bernoulli's equation, $p/\rho=C-\frac{1}{2}V^2-gy$, gives

$$\frac{p}{\rho}=C'+\frac{g\epsilon}{k}i\left[(1-\tfrac{1}{2}\mu^2)-\frac{kc^2}{g\nu^2}(1+\tfrac{1}{2}\mu^2)\right][e^{-i(\tau+\bar{\alpha})}-e^{i(\tau+\alpha)}]$$

where $\nu=1+\mu^2$, $\mu=\epsilon e^{-b}$.

If one takes b_0 as corresponding to the free-surface trajectory and sets the first expression in brackets equal to zero in order to satisfy (within the approximation allowed) the usual free-surface condition, $p=$const., one may obtain from the resulting equation $c^2=gk^{-1}(1+\mu_0^2)+$terms of higher order in ϵ. Determination of the amplitude A from the equation for y gives finally the known Stokes formula $c^2=(1+4\pi^2A^2/\lambda^2)g\lambda/2\pi$, etc. For the case of the periodic pressure distribution, write the expression for p/ρ as $p/\rho=C'+\beta\sin(\tau+a)$. The relation between c, λ, and A is no longer determined by the problem, but the author takes $c^2=g\lambda/2\pi$. This gives $\beta=g\mu_0^2/k+$terms of higher order in ϵ, which gives a relation between the period T, c, β, and k. The wave profile is also determined.

J. V. Wehausen (Providence, R. I.).

Longuet-Higgins, M. S. Mass transport in water waves. Philos. Trans. Roy. Soc. London. Ser. A. **245**, 535–581 (1953).

The mass-transport velocity for a progressive wave in an inviscid liquid of finite depth is negative (with respect to the wave direction) at the bottom and increases monotonically to a positive value at the surface [cf. the author, Proc. Cambridge Philos. Soc. **49**, 552–560 (1953); these Rev. **15**, 173]. This result is, however, in disagreement with some experiments in water which indicate a positive velocity at the bottom and a negative, but increasing, velocity near the surface. In the present paper the author develops the theory of mass transport for a viscous fluid under the assumptions that the velocity is periodic in time and that the motion can be expressed as a perturbation of a state of

rest (e.g., the velocity can be expressed as a series in a small parameter ϵ: $u=\epsilon u_1+\epsilon^2 u_2+\cdots$). The mass-transport velocity is defined and the necessary equations for the successive approximations derived from the equations of motion. A boundary-layer approximation method of solution is developed for the cases when the velocities on the boundary are prescribed and when the stresses are prescribed. The method is applied to both progressive and standing waves in water of uniform depth, giving results in qualitative agreement with the experimental results mentioned above. For the wealth of detailed discussion of the nature of the approximations one must refer to the paper itself.

J. V. Wehausen (Providence, R. I.).

Dmitriev, A. A. Waves on the surface of a viscous fluid which are generated by a pulsating source. Izvestiya Akad. Nauk SSSR. Ser. Geofiz. **1953**, 335–345 (1953). (Russian)

The author derives expressions for the surface waves due to a submerged source of intensity $-Q\cos\sigma t$. Inertial terms in the equations of motion are neglected (the resulting equations and boundary conditions are precisely those given in Lamb's Hydrodynamics, 6th ed., Cambridge, 1932, §349, although the author credits them to an unavailable 1941 report of Sretenskiĭ). The equation of the free surface is

$$\eta=\mathrm{Re}\,\frac{Q\exp i\sigma t}{\pi}\,\frac{\sigma}{g}\int_0^\infty \frac{1-2i\chi^2}{4\epsilon[i\chi^2(\chi-(i+\chi^2)^{1/2})-\chi^2]+i(\chi-\epsilon)}\times\exp(-h\chi)\cos\chi\xi\,d\xi$$
$$=\frac{Q\sigma}{g}(1+100\epsilon^4)^{1/2}\exp(-h\epsilon-4\epsilon^3\xi)\times\cos(\sigma t-\epsilon\xi+4\epsilon^3 h-\arctan 10\epsilon^2)+\cdots,$$

where $h(\nu/\sigma)^{1/2}$ is the depth of submersion of the source, $\xi(\nu/\sigma)^{1/2}=x$ is measured horizontally, and $\epsilon=\sigma^{1/2}\nu^{1/2}/g$. The result is applied to a discussion of the damping of surface waves and of the formation of waves by a wave maker.

J. V. Wehausen (Providence, R. I.).

Rusanov, B. V. Slow unsteady flow of a viscous fluid about a circular cylinder. Doklady Akad. Nauk SSSR (N.S.) **89**, 983–986 (1953). (Russian)

L'auteur se propose de construire dans le plan Oxy (rapporté aux coordonnées polaires r, θ) l'écoulement (régi par les équations linéaires de Navier-Stokes) du liquide visqueux indéfini autour de l'obstacle circulaire $r=a$. Le champ des vitesses $\mathbf{V}(r,\theta,t)$ vérifie par hypothèse les conditions suivantes: (1) $\mathbf{V}(a,\theta,t)=0$ pour $0\leqq t\leqq\infty$. (2) Le champ initial $\mathbf{V}(r,\theta,0)$ est symétrique par rapport à Ox (hypothèse non essentielle d'après l'auteur), différentiable deux fois en r et θ, lipschitzien en θ (exposant $\alpha>\frac{1}{2}$), tel que $|\partial\mathbf{V}/\partial r|=O(1/r)$ pour $r=\infty$. D'après un résultat de Dolidze [Soobščeniya Akad. Nauk Gruzin. SSR **8**, 11–18 (1947); ces Rev. **13**, 878], la solution du problème aux limites ainsi posé, est unique. L'auteur l'exprime au moyen de séries de Fourier en θ, dont les coefficients sont donnés sous formes d'intégrales définies. L'auteur déduit de là l'explication du paradoxe de Stokes (non-existence d'un écoulement permanent, plan autour d'un cylindre de révolution). La même méthode s'applique aux mouvements lents d'un cylindre dans un liquide indéfini.

J. Kravtchenko (Grenoble).

Rusanov, B. V. Slow unsteady flow of a viscous fluid about a sphere. Doklady Akad. Nauk SSSR (N.S.) **90**, 41–44 (1953). (Russian)

La méthode du mémoire analysé ci-dessus est étendue, sous des hypothèses semblables, à l'écoulement à symétrie sphérique d'un liquide visqueux autour d'une sphère animée d'un mouvement de translation rectiligne. Toutefois, l'auteur semble ignorer les travaux de M. Villat [Leçons sur les fluides visqueux, Gauthier-Villars, Paris, 1943, pp. 192–224] qui conduisent, par des voies entièrement différentes, à une solution un peu plus générale. Il est vrai que Villat ne démontre pas l'unicité de sa solution. *J. Kravtchenko.*

Tyabin, N. V. Motion of a sphere in a viscous-plastic liquid dispersive system. Doklady Akad. Nauk SSSR (N.S.) **88**, 57–60 (1953). (Russian)

L'auteur utilise les équations dynamiques du milieu viscoplastique qui lui sont dues pour résoudre le problème du mouvement d'une sphère solide animée d'un mouvement de translation rectiligne dans un liquide indéfini. Les forces d'inertie sont négligées vis-à-vis des forces de viscosité.

J. Kravtchenko (Grenoble).

Tomotika, S., and Yosinobu, H. The drag on an elliptic cylinder moving in a viscous liquid at small Reynolds numbers. Mem. Coll. Sci. Univ. Kyoto Ser. A. **27**, 11–22 (1953).

Following Sidrak's method [Proc. Roy. Irish Acad. Sect. A. **53**, 65–81 (1950); these Rev. **12**, 137], the authors repeat Sidrak's calculations and obtain results which agree with those arrived at by different methods by Tomotika and Aoi [Quart. J. Mech. Appl. Math. **6**, 290–312 (1953); these Rev. **15**, 174] and by Tomotika [see the paper reviewed below]. For those interested, Sidrak's later paper [Proc. Math. Phys. Soc. Egypt **4**, no. 4, 17–27 (1953); these Rev. **15**, 174] should also be consulted. *Y. H. Kuo.*

Tomotika, S. A note on the skin friction of a flat plate to Oseen's approximation. Mem. Coll. Sci. Univ. Kyoto Ser. A. **27**, 1–9 (1953).

In this note the author makes an independent calculation of the drag exerted on a flat plate at low Reynolds numbers Re by the method of Bairstow, Cave and Lang [Philos. Trans. Roy. Soc. London. Ser. A. **223**, 383–432 (1923)]. The expansion in Reynolds number has been carried out to Re^4 and agrees with the results of Tomotika and Aoi and Tomotika and Yosinobu [see the preceding review].

Y. H. Kuo (Ithaca, N. Y.).

Ševelev, Ya. V. Poiseuille flow in an unsymmetric annular gap. Analogy with torsion of a beam. Doklady Akad. Nauk SSSR (N.S.) **91**, 35–38 (1953). (Russian)

Let C_0 be a simple closed curve containing several other such curves C_i. Let these contours be taken as the normal cross-section of a cylindrical pipe with cylindrical inserts. The various pipes will be assumed to move lengthwise with velocities v_i. A fluid of viscosity μ within C_0 and outside the C_i is assumed to be in steady laminar motion parallel to the pipes subject to a pressure gradient dp/dl. To find the velocity $v(x,y)$ at any point of the cross-section, one must solve $\Delta v+\mu^{-1}dp/dl=0$ with the boundary conditions $v=v_i$ on C_i. The author shows how this problem is related to the problem of torsion of a rod with longitudinal holes [cf., e.g., Sokolnikoff, Mathematical theory of elasticity, McGraw-Hill, New York, 1946, p. 191 ff.], and how one may proceed from a solution of one of these problems to that of the other. This is illustrated by using Macdonald's solution [Proc. Cambridge Philos. Soc. **8**, 62–68 (1893)] for the torsion of a circular cylinder with an excentric circular hole to obtain the solution for the corresponding viscous flow problem. Graphs are given for the case $v_0=v_1=0$. This problem had

been treated incorrectly by M. D. Kuznecov [same Doklady (N.S.) **85**, 715–717 (1952); these Rev. **14**, 424].

J. V. Wehausen (Providence, R. I.).

Loĭcyanskiĭ, L. G. Propagation of a whirling jet into an infinite space filled with the same fluid. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 3–16 (1953). (Russian)

Une veine de liquide visqueux de révolution s'échappe d'un orifice dans un espace illimité, rempli du même liquide. L'auteur se propose d'étudier le régime stationnaire, supposé établi dans tout l'espace. Ce problème a déjà été étudié par Rumer [même journal **16**, 255–256 (1952); ces Rev. **13**, 791] et Landau, mais ces auteurs ont négligé les vitesses transversales qui entraînent la "torsion" de la veine. L'auteur aborde le cas de la faible "torsion". Soient: x, r, θ, les coordonnées cylindro-polaires de l'espace (Ox étant pris suivant l'axe de révolution de la veine, dont la source est à l'abscisse $x=0$); u, v, w les composantes de la vitesse suivant les lignes coordonnées. On pose: $\eta=r/x\nu^{1/2}$. Dans le cas laminaire, l'auteur simplifie d'abord les équations de Navier, en supposant v et w petits, puis il cherche les solutions u, v, w du problème aux limites, auquel il aboutit, sous forme de séries du type: $\sum_{n=1}^{\infty}a_n(\eta)/x^n$ (non convergents dans le voisinage de l'origine). Les inconnues $a_n(\eta)$ sont solutions d'un système compliqué; l'auteur calcule approximativement les deux premiers termes des développements; on peut tirer de là quelques conclusions qualitatives intéressantes quant à l'allure de l'écoulement.

Lorsque l'écoulement devient turbulent, une série d'hypothèses qui semblent assez naturelles et dont les plus audacieuses concernent la forme, à priori, du tenseur des efforts turbulents, permet de se ramener, au point de vue de l'intégration, au cas laminaire. A noter que suivant l'auteur le contrôle expérimentale de cette théorie a été satisfaisant.

J. Kravtchenko (Grenoble).

Boreli, Mladen. Sur une solution rigoureuse d'un problème d'écoulement plan des nappes souterraines. C. R. Acad. Sci. Paris **237**, 132–135 (1953).

Plane steady-state flow of liquid into a drain carrying it at constant pressure is studied. Contrasting with previous experimental work of the same author [same C. R. **235**, 785–786 (1952)], conformal mapping is used to obtain the result in this case, though only a short derivation and a simple sketch are included. *R. E. Gaskell.*

Pack, D. C. Laminar flow in an axially symmetrical jet of compressible fluid, far from the orifice. Proc. Cambridge Philos. Soc. **50**, 98–104 (1954).

The author shows that when the Prandtl number is unity, the equations of motion for the flow in an axially symmetric jet of a compressible and viscous fluid can be reduced to one equation for the stream function in terms of both the old and new radial distances. If the old radial distance multiplied by density is equal to the new radial distance and the viscosity is constant, the equation becomes of the same form as in the incompressible case. It is shown, moreover, that for Mach number less than unity the error in this approximation is small when the distance from the orifice is large.

Y. H. Kuo (Ithaca, N. Y.).

Maekawa, T. One method of solving a compressible laminar boundary layer. J. Sci. Hiroshima Univ. Ser. A. **17**, 83–110 (1953).

By a series of transformations of variables, the equations of motion for steady-two-dimensional, compressible, laminar boundary-layer flow, where the Prandtl number is 1 and there is no heat transfer, are reduced to the forms used by von Kármán and Millikan [NACA Rep. no. **504** (1934)] in the corresponding incompressible case. These final forms involve certain neglects for (a) the region near the wall and (b) the outer region. The transformations appropriate to these two regions are not identical. The method of solution is the same as used by von Kármán and Millikan, and a certain extension is suggested. The effects of compressibility in the inner and outer regions, which are different, are discussed. Special cases treated in detail include (1) linear velocity gradient in the potential flow, (2) linear Mach number gradient, (3) linear "effective velocity gradient" in the outer region of boundary layer flow. In each case the separation criterion is worked out.

W. R. Sears (Ithaca, N. Y.).

Nekrasov, K. P. Motion of a cylindrical body in a fluid, according to the theory of vanishing viscosity (two-dimensional steady flow). Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 17–32 (1953). (Russian)

La théorie linéaire de la viscosité évanescente [cf. H. Villat, Leçons sur l'hydrodynamique, Gauthier-Villars, Paris, 1929, pp. 151–296] a permis à Oseen et à Zeilon de traiter, en première approximation, le problème de l'écoulement d'un liquide visqueux indéfini, au repos à l'infini, autour d'un solide animé d'un mouvement de translation rectiligne et uniforme; la solution de Zeilon présente, dans le cas particulier du régime plane, des singularités qui en limitent l'intérêt physique. Villat indique la voie possible pour surmonter ces difficultés: la révision des conditions imposées à priori à certains éléments de la solution. L'auteur, justement, propose un schéma du mouvement légèrement différent du schéma classique de Oseen. Il parvient aussi à construire une solution du problème qui respecte la continuité du champ des vitesses; le vecteur tourbillon garde, d'autre part, une intensité finie aux points de détachement du sillage. *J. Kravtchenko* (Grenoble).

***Tifford, Arthur N., and Chu, Sheng To. The laminar boundary layer on spinning axisymmetric bodies.** Proceedings of the Third Midwestern Conference on Fluid Mechanics, University of Minnesota, March 23, 24, and 25, 1953, pp. 579–592. University of Minnesota, Minneapolis, Minn., 1953. $6.00.

The equations of motion and the energy equation are written out for the case described by the title, assuming constant density and viscosity. Their limitations are discussed. The analogies between tangential velocity component and temperature and between spin torque and heat transfer, for Prandtl number equal to 1, are pointed out. The meaning of the latter analogy for other Prandtl numbers, and the effects of large spin are discussed.

W. R. Sears (Ithaca, N. Y.).

Shen, S. F. Calculated amplified oscillations in the plane Poiseuille and Blasius flows. J. Aeronaut. Sci. **21**, 62–64 (1954).

Krasil'nikov, V. A., and Tatarskiĭ, V. I. Dispersion of sound in a turbulent flow. Doklady Akad. Nauk SSSR (N.S.) **90**, 159–162 (1953). (Russian)

***Hama, Francis R. The spectrum equation of two-dimensional isotropic turbulence.** Proceedings of the Third Midwestern Conference on Fluid Mechanics, University of Minnesota, March 23, 24, and 25, 1953, pp. 427–433. University of Minnesota, Minneapolis, Minn., 1953. $6.00.

L'auteur part de l'équation qui exprime $F_1(k)$, transformée de Fourier de la fonction de corrélation longitudinale $f(r)$ dans un écoulement plan incompressible isotrope, à l'aide de la fonction spectrale bidimensionnelle $F^*(k)$. Il commence par établir la formule inverse, exprimant $F^*(k)$ à l'aide de $F_1(k)$, puis il établit l'équation d'évolution de $F^*(k)$ en fonction du temps, et la compare avec l'équation analogue à trois dimensions. *J. Bass* (Chaville).

***v. Krzywoblocki, M. Z. On the decay of turbulence in compressible fluids in terms of vorticity.** Proceedings of the Third Midwestern Conference on Fluid Mechanics, University of Minnesota, March 23, 24, and 25, 1953, pp. 413–425. University of Minnesota, Minneapolis, Minn., 1953. $6.00.

L'auteur considère un fluide turbulent compressible. Il donne en appendice une liste de développements limités d'un grand nombre d'expressions moyennes (corrélations) attachées à la vitesse, la pression, la densité, la température. Il établit ensuite, pour l'énergie cinétique, le tourbillon quadratique moyen, l'enthalpie, et diverses longueurs caractéristiques, des équations de décroissance analogues aux équations de Taylor et de Kármán-Howarth pour l'énergie cinétique et la corrélation longitudinale dans un fluide incompressible. *J. Bass* (Chaville).

Lighthill, M. J. On the energy scattered from the interaction of turbulence with sound or shock waves. Proc. Cambridge Philos. Soc. **49**, 531–551 (1953).

If a plane harmonic sound wave,

$$V_i = ea\cos[\kappa(x_1 - at)]\delta_{i1}$$

of velocity a and frequency $\kappa a/2\pi$, travelling in the x_1-direction is incident on a turbulent fluid (mean density ρ_0), then the scattering of acoustic energy by the turbulent velocities v_i' can be regarded as arising from a distribution of quadrupoles of strength

$$T_{ij} = \rho_0 ea\cos[\kappa(x_1 - at)](v_i'\delta_{j1} + v_j'\delta_{i1}),$$

per unit volume in the medium. Using the expression [cf. Lighthill, Proc. Roy. Soc. London. Ser. A. **211**, 564–587 (1952); these Rev. **13**, 879]

$$\rho - \rho_0 = \frac{1}{4\pi a^4}\frac{x_i x_j}{x^3}\int \frac{\partial^2}{\partial t^2} T_{ij}\left(y, t - \frac{|x-y|}{a}\right)dy,$$

for the radiation field resulting from a quadrupole distribution, T_{ij}, the author finds that, on certain suitable approximations, the intensity of the scattered energy I_s crossing unit area at x is given by

$$I_s = \frac{2\pi I\kappa^4}{a^2}\frac{x_i x_j x_1^2}{x^6}F_{ij}(k),$$

where

$$F_{ij}(k) = \frac{1}{8\pi^3}\int\int \overline{v_i'(y)v_j'(z)}\cos[k\cdot(y-z)]dydz,$$

$I(=\frac{1}{2}\rho_0 a^3 e^2)$ is the intensity of the incident wave and $k_i = \kappa[(x_i/x) - \delta_{i1}]$. For homogeneous turbulence

$$\overline{v_i'(y)v_j'(z)} = R_{ij}(|z-y|)$$

and the corresponding spectrum tensor

$$\Gamma_{ij}(k) = \frac{1}{8\pi^3}\int R_{ij}(y)e^{-ik\cdot y}dy,$$

is given by

$$\Gamma_{ij}(k) = \frac{E(k)}{4\pi k^4}(k^2\delta_{ij} - k_i k_j),$$

where $E(k)$ denotes the spectrum of turbulence. For this special case we deduce that the rate p_s at which energy is scattered per unit volume of isotropic turbulence is

$$p_s = \frac{\pi\kappa^3}{a^2}I\int_0^{2\kappa}\frac{E(k)}{k}M(k/\kappa)dk$$

where $k = 2\kappa\sin\frac{1}{2}\theta$ and $M(x) = (1-\frac{1}{4}x^2)^2(1-\frac{1}{4}x^2)$. For $\kappa \gg k$, the foregoing expression can be approximated by setting $M=1$ and we have

$$p_s = \frac{\pi\kappa^2}{a^2}I\int_0^\infty E(k)\frac{dk}{k} = 2\kappa^2 L_1\frac{\overline{v_1'^2}}{a^2}I,$$

where $L_1 = (\pi/2\overline{v_1'^2})\int_0^\infty E(k)k^{-1}dk$ is Taylor's 'macroscale' as usually defined.

When a sound pulse represented by $V_i = af(x_1 - at)\delta_{i1}$ is incident, the author shows that the mean total scattered energy $G(s)$ is given by

$$G_s = \int_0^\infty G(\kappa)d\kappa\left[\frac{2\pi\kappa^4}{a^2}\frac{x_i x_j x_1^2}{x^6}F_{ij}(k)\right]$$

where $G(\kappa)$ is the spectrum of the energy pulse: i.e., $G(\kappa) = 4\pi\rho_0 a^2 F(\kappa)F(-\kappa)$ where $F(\kappa)$ is the Fourier transform of $f(r)$.

The author discusses the foregoing results from the point of view of various applications. *S. Chandrasekhar.*

Mackie, A. G. One-dimensional unsteady motion of a gas initially at rest and the dam-break problem. Proc. Cambridge Philos. Soc. **50**, 131–138 (1954).

The determination of one-dimensional isentropic flow with velocity $u=0$ and speed of sound $c = (\gamma-1)r$ at $x = \xi(r)$, $t=0$ can be reduced to solving

$$\partial^2 w/\partial r\partial s + N(r+s)^{-1}(\partial w/\partial r + \partial w/\partial s) = 0$$

with $\partial w/\partial r = -\partial w/\partial s = \xi(r)$ on $r=s$, where

$$N = (3-\gamma)/(2\gamma-2),$$

and r, s are Riemann invariants. E. T. Copson [Proc. Roy. Soc. London. Ser. A. **216**, 539–547 (1953); these Rev. **14**, 816] has shown that if ξ is even and analytic and $N>0$ then

$$2\pi i w = \int_C z^{2N}\varphi(z)(z-r)^{-N}(z+s)^{-N}dz,$$

where $z^{2N}\varphi(z) = 2\int_0^z (z^2-r^2)^{N-1}\xi(r)rdr$ and C is a closed curve surrounding a cut containing the points 0, r, $-s$. By exploiting properties of beta and gamma functions the author shows that w has the same form for odd ξ, except that C now consists of two contours starting at $z=0$ on the upper side of the cut and proceeding in opposite directions around the ends of the cut to $z=0$ on the lower side. He also shows that for $\xi = -A^2 r$, $0 \le r \le R$, and for all $N>0$ all characteristics $r=$ const. that emanate from points ξ, 0 in the xt-plane pass through $x=0$, $t = A^2 B(N+1, \frac{1}{2})/(2\gamma-2)$, previously shown for integers N by D. C. Pack [Proc. Cambridge Philos. Soc. **49**, 493–497 (1953); these Rev. **14**, 1032]. The results are interpreted for a gas cloud expanding into a

vacuum and for the dam-break problem by the hydraulic analogy with $\gamma=2$. *J. H. Giese* (Havre de Grace, Md.).

Sagomonyan, A. Ya. Investigation of the linearized equation of self-similar unsteady motion of a fluid. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk **1952**, no. 9, 3–8 (1952). (Russian)

L'auteur étudie l'équation des ondes cylindriques:

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=\frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2}. \tag{1}$$

Il explicite la forme de (1) en variables: $\xi=x/t$, $\eta=y/t$, et cherche les solutions de la transformée (E) et (1) du type: $\varphi=t\varphi_1(\xi,\eta)$. (E) est hyperbolique (elliptique) dans le domaine $\mathfrak{D}_H$ (ou $\mathfrak{D}_E$): $\xi^2+\eta^2>a^2$ (ou $\xi^2+\eta^2<a^2$). Les deux familles de caractéristiques de E dans $\mathfrak{D}_H$ se déterminent aisément; le long de chaque courbe caractéristique il existe une relation linéaire entre $\partial\varphi/\partial x$ et $\partial\varphi/\partial y$. Un artifice analogue à celui de Tchaplyguine permet d'exprimer les solutions de (E) dans $\mathfrak{D}_E$ au moyen d'une fonction analytique de variable complexe: d'où possibilité de résoudre divers problèmes aux limites pour (E) dans $\mathfrak{D}_E$. Les solutions particulières, formées par cette méthode, sont, aux dires de l'auteur, dépourvues de discontinuités, qui apparaissent nécessairement dans les écoulements stationnaires à zones sur- et subsoniques. *J. Kravtchenko.*

Šul'gin, D. F. Flow about a composite profile of variable permeability. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 285–292 (1953). (Russian)

Envisageons un écoulement permanent plan du liquide, uniforme à l'infini, incliné sur l'axe Ox. Sur une longueur finie de cet axe se trouve réparti un nombre fini d'obstacles rectilignes. Tchaplyguine [Oeuvres, t. II, Gostehizdat, Moscow-Leningrad, 1948, pp. 431–471; ces Rev. **14**, 609] a formé la fonction caractéristique du régime. L'auteur par un passage à la limite, étend ces conclusions au cas où le nombre de segments augmenterait indéfiniment alors que la longueur de chacun d'eux tendait vers zéro; la densité des parties pleines étant inférieure à 1. Dans la deuxième partie du mémoire, l'auteur reprend la configuration étudiée par Tchaplyguine, mais en supposant que les obstacles sont attaqués par un jet gazeux subsonique. Moyennant quelques hypothèses empruntées à la théorie des ailes minces, le problème est ramené à une équation intégrale singulière que l'on peut résoudre. *J. Kravtchenko* (Grenoble).

Volt, S. S. Propagation of an initial condensation in a viscous gas. Doklady Akad. Nauk SSSR (N.S.) **88**, 221–224; correction **90**, 124 (1953). (Russian)

Un problème de propagation d'onde de condensation, au sein d'une masse de gaz emplissant l'espace, conduit au problème de Cauchy pour l'équation:

$$\frac{\partial^2 s}{\partial t^2}=c^2\Delta s+\frac{4}{3}\nu\frac{\partial\Delta s}{\partial t}$$

où ν est la viscosité cinématique du gaz, c la vitesse du son dans le gaz parfait, avec les conditions aux limites:

$$s(x,y,z,0)=\omega_1(x,y,z);\quad\left[\frac{\partial s(x,y,z,t)}{\partial t}\right]_{t=0}=\omega_2(x,y,z),$$

ω_1 et ω_2 étant des données, d'ailleurs nulles à l'extérieur de la zone G, où se trouve localisée la perturbation à l'instant $t=0$. Le théorème de Petrovsky permet d'établir que le problème ci-dessus est bien posé. L'expression explicite de la solution est très compliquée, mais elle se simplifie beaucoup si G est une sphère centrée sur l'origine et si les données initiales sont à symétrie sphérique; on peut alors discuter assez aisément les particularités du phénomène.

J. Kravtchenko (Grenoble).

Kofink, W., und Vollmer, Th. Der gegabelte Verdichtungsstoss in Luft. Z. Angew. Math. Mech. **33**, 73–88 (1953). (English, French, and Russian summaries)

This paper is a continuation of an earlier series of papers by Kofink [Ann. Physik (6) **9**, 200–212, 401–405 (1951); **10**, 200 (1952); these Rev. **13**, 296, 702, 793] on an algebraic analysis of triple shock configurations in a steady supersonic stream. Extensive calculations have been made and tables and curves are presented depicting possible triple shock configurations for air ($\gamma=1.405$) in terms of various parameters. Particular emphasis is placed on special and limiting configurations and relationships for arbitrary γ are derived for these cases. In addition, the authors give an algebraic treatment of the "heart curve" method of A. Weise for the study of triple shock configurations together with an analysis of the use of this method in considering the number of possible solutions for each set of values of the various parameters. *P. Chiarulli* (Providence, R. I.).

***Burgers, J. M. Nonuniform propagation of plane shock waves.** Proceedings of Symposia in Applied Mathematics, vol. IV, Fluid dynamics, pp. 101–108. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. $7.00.

This is essentially a summary paper of some previously published work by the author and others on the problem of the title. The treatment uses a Lagrangian description of the one-dimensional motion in the wake of a shock wave which propagates non-uniformly due to such causes as an original non-homogeneous state of the gas, the presence of an external force such as gravity, etc. The equations of motion are obtained and three procedures for finding solutions are considered. These are a double series development in the Lagrangian variable and time; an approximate method introduced by Brinkley and Kirkwood which makes use of an empirical coefficient involved in the energy of the shock wave; and the construction of explicit solutions for particular cases in which the original data is suitably prescribed. *P. Chiarulli* (Providence, R. I.).

Marchal, Raymond. Sur les conditions d'apparition des ondes de choc dans les écoulements permanents plans. C. R. Acad. Sci. Paris **237**, 1392–1394 (1953).

Pérès, Joseph. Remarques sur la Note de M. Marchal. C. R. Acad. Sci. Paris **237**, 1394–1395 (1953).

Ferrari, Carlo. Sul potenziale di un filetto vorticoso in corrente ipersonica linearizzata. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **86**, 3–8 (1952).

The author discusses the field of flow due to a line vortex in linearised supersonic flow. Transforming a formula due to A. Robinson [Quart. J. Mech. Appl. Math. **1**, 408–432 (1948); these Rev. **10**, 410] he shows that the same field of flow is produced by a particular distribution of supersonic doublets over a surface which is bounded by the line vortex. Unlike in subsonic flow, the axes of the doublets are not normal to the surface but transversal to it. The field of flow due to a circular line vortex in a plane which is normal to the direction of the free stream, is calculated as an example.

A. Robinson (Toronto, Ont.).

Hasimoto, Zirō. Plane jet of gas emitted from a nozzle with supersonic velocity. J. Phys. Soc. Japan **8**, 394–399 (1953).

The author considers the problem of the periodic character of a uniform supersonic plane gas jet issuing steadily from a nozzle into an external gas at rest when the ratio of exist pressure to the pressure in the external medium is near unity. The approximate pressure-density relationship (equivalent to using a pressure-density curve coinciding at a typical point up to curvature with the corresponding curve for the real gas) introduced by Christianovich [Akad. Nauk SSSR. Prikl. Mat. Mech. **11**, 215–222 (1947) = Amer. Math. Soc. Translation no. **10**, 3–16 (1950); these Rev. **9**, 390; **11**, 272] and the author [Mem. Coll. Sci. Univ. Kyoto Ser. A. **26**, 193–206 (1950)] is used and as a result the hodograph equations take on a particularly simple form. A closed form solution of the interaction of two two-dimensional expansion waves is found and from this result an expression for the "wave length" of the jet is obtained. A "wave length" is computed for conditions corresponding to the one experimental point that seems readily available and it differs from the experimental value by 0.2% whereas the latest result based on a linearized theory differs by 9%. On the basis of this comparison the author concludes that since his solution takes account of non-linear terms it is much more accurate than the linear theory based on the more accurate pressure-density relationship. *P. Chiarulli.*

Ferri, Antonio. The linearized characteristics method and its application to practical nonlinear supersonic problems. NACA Rep. no. **1102**, ii+18 pp. (1952).

Issued earlier as NACA Tech. Note no. **2515** (1951); these Rev. **13**, 400.

Krzywoblocki, M. Z. E. Bergman's linear integral operator method in the theory of compressible fluid flow. B. Super- and transonic flow. Österreich. Ing.-Arch. **7**, 336–370 (1953).

In this second part of his work on Bergman's linear integral operator method [for part I see same journal **6**, 330–360 (1952); these Rev. **14**, 510] the author discusses supersonic and transonic flow. At first, the operator obtained by the use of Riemann's function is thoroughly discussed as well as the question of the domain of convergence. In the section on transonic flow the application of the integral operators of the first and second kind in the case of the exact and the simplified compressibility equation is shown. *Author's summary.*

Spreiter, John R. On alternative forms for the basic equations of transonic flow theory. J. Aeronaut. Sci. **21**, 70–72 (1954).

Dodd, K. N. A case of gaseous accretion. Proc. Cambridge Philos. Soc. **49**, 486–492 (1953).

This paper is devoted to a consideration of axially symmetric isothermal flows of gas past a gravitating point-sink for values of the undisturbed velocity which are entirely supersonic. For the velocity potential ϕ the differential equation to be solved in cylindrical polar co-ordinates is

$$(1)\quad (\phi_z^2-c^2)\phi_{zz}+2\phi_r\phi_z\phi_{rz}+(\phi_r^2-c^2)\phi_{rr} = -\mu[(r\phi_r+z\phi_z)/(r^2+z^2)^{3/2}]+c^2\phi_r/r,$$

where μ is a constant and c $(=\sqrt{(dp/d\rho)})$ denotes the constant velocity of sound. It is shown that the characteristics of this hyperbolic equation are given by

$$(2)\quad dr/dz=\tan(\theta\pm m)$$

where θ $(=\tan^{-1}u/v;\ u=\phi_z,\ v=\phi_r)$ is the inclination of the stream line to the z-axis and m $(=\sin^{-1}c/\sqrt{(u^2+v^2)})$ is the Mach angle. Further along the characteristics

$$(3)\quad dm\pm\tan^2 m\,d\theta + \frac{\tan^2 m\sin m}{\cos(\theta\pm m)}\left[\frac{\sin\theta}{r}-\frac{\mu}{c^2}\frac{r\sin\theta+z\cos\theta}{(r^2+z^2)^{3/2}}\right]dz=0.$$

A numerical method for solving (1) based on integrating along the characteristics using eqs. (2) and (3) is described and illustrated. The solution of the hydrodynamic problem is used to determine the stagnation point on the axis of flow. Indications are found that shocks may appear in flows of the type considered. *S. Chandrasekhar.*

Heinz, C. Überschallströmungen um schlanke Drehkörper. Z. Angew. Math. Mech. **33**, 306–308 (1953).

The author develops a method for the reduction of the Volterra integral equations

$$\int_0^{\lambda_0} f'_r[x-(M^2-1)^{\frac{1}{2}}k(x)\cosh\lambda]\cosh^r\lambda\,d\lambda=S_r(x),\quad r=1,2,$$

obtained originally by von Kármán and Moore in their linearized treatment of the supersonic flow past a slender body of revolution [Trans. A.S.M.E. **APM-54**, 303–310 (1932)], to differential equations, thereby claiming to reduce the labor of a numerical solution. However, since an explicit solution to the original problem, after consistent linearization, has been given by Lighthill [Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2003** (1945); these Rev. **8**, 108; also L. Howarth et al., Modern developments in fluid dynamics, high speed flow, v. 1, Oxford, 1953, p. 307] the author's results would appear to be of limited interest in this connection. *J. W. Miles* (Los Angeles, Calif.).

Miles, John W. On the low aspect ratio oscillating rectangular wing in supersonic flow. Aeronaut. Quart. **4**, 231–244 (1953).

The Laplace transform of the lift distribution on an oscillating rectangular wing in a supersonic flow is obtained by separating the linearised equation for the velocity potential in elliptic (cylindrical) co-ordinates. The results for the case of no spanwise distortion are expanded in ascending powers of the aspect ratio in order to compare with the slender-body theory, and the longitudinal stability derivatives are calculated. It is found that at either supersonic or transonic speeds single-degree-of-freedom instability in pitch is impossible insofar as the fourth power of the aspect ratio is neglected. *C. C. Chang* (College Park, Md.).

Lomax, Harvard, Heaslet, Max. A., Fuller, Franklyn B., and Sluder, Loma. Two- and three-dimensional unsteady lift problems in high-speed flight. NACA Rep. no. **1077**, ii+55 pp. (1952).

Supersedes Lomax, Heaslet, and Fuller, NACA Tech. Note no. **2387** (1951) [these Rev. **13**, 181] and Lomax, Heaslet, and Sluder, ibid. no. **2403** (1951), and contains material from Lomax, Heaslet, and Fuller, ibid. no. **2256** (1950) [these Rev. **12**, 767]. In Tech. Note no. **2256** (among other problems) the indicial motion of a slender triangular wing in supersonic flight was analyzed by neglecting the streamwise derivatives in the wave equations referred to

coordinates at rest in the medium, thereby rendering the boundary conditions time dependent, and the reviewer criticized the approximation as inconsistent. In the present report it is shown that the same approximation can be referred to a coordinate system fixed in the wing. The resulting boundary value problem, viz.

$$\phi_{tt}-\phi_{xx}-\phi_{zz}=0 \tag{1}$$

$$\phi_z=1(t), \quad z=0, \quad |y|<mx \tag{2}$$

$$\phi=0 \quad z=0, \quad |y|>mx, \tag{3}$$

where 1 denotes the Heaviside step function, is solved exactly for small t and approximately for large t by posing an elliptic load distribution over the span and solving the resulting, one-dimensional integral equation numerically.

The reviewer notes that (1)–(3) also represent the classical problem of diffraction by a ribbon, and the methods of attack originally proposed by K. Schwarzschild [Math. Ann. 55, 177–247 (1902)] and B. Sieger [Ann. Physik (4) 27, 626–664 (1908)] are applicable and yield more accurate results (than the authors' approach) with less numerical labor, as exemplified by the reviewer's analyses of the low aspect ratio, rectangular wing [see the paper reviewed above] and impulsive motion of a ribbon [Quart. J. Mech. Appl. Math. 6, 129–140 (1953); these Rev. 15, 73].

The report includes a large number of graphs giving the indicial responses of two-dimensional wings at Mach numbers from 0 to 2, for supersonic leading edge triangular wings in forward and reverse flight, and for a slender triangular wing for which the product of the tangent of the semi-apex angle and the Mach number is 1/8. Also included, for purposes of comparison, are the reviewer's results for the indicial motion of a supersonic rectangular wing [Naval Ordnance Test Station, Inyokern, Calif., Tech. Memo. RRB-27 (1949); J. Aeronaut. Sci. 17, 647–652 (1950); these Rev. 12, 217, 369]. *J. W. Miles* (Los Angeles, Calif.).

Queney, P. Phénomènes de résonance et d'instabilité dans les écoulements barotropes, avec applications aux ondes atmosphériques de grande échelle. Ann. Géophysique 9, 185–226 (1953).

A fluid of constant density moves in a plane so that its velocity is parallel to the x-axis of rectangular coordinates (x, y) and varies linearly with y. The component of vorticity perpendicular to the plane is constant "following the motion". Perturbations of this flow are considered in which the perturbation of the vorticity, q, is still conserved, except that discontinuities of q are allowed on one or more lines $y=$constant. Parallel currents, in which the velocity varies with different constant gradients with respect to y, are considered, but there are no discontinuities of velocity at the boundaries of the currents. When there are two currents, and discontinuities of q occur on $y=0$ (the boundary between the currents) and on another line $y=h$, waves in the currents result of which one type, the "resonance waves", have amplitudes that increase linearly with the time. These results are generalized to the case when there are more than two parallel currents. The results can also be adapted to meridional currents on a surface of revolution, in particular a sphere, by means of a conformal transformation of coordinates of Mercator type. Hence the perturbations of Rossby currents at the barotropic 12 km. level in the atmosphere can be studied. The resonance waves are shown to be in good agreement with cyclonic waves. The sequence of events suggested is: (1) a perturbation of vorticity in the 12 km. region is produced through some disturbance originating in the lower troposphere; (2) a system of resonance waves develops in the 12 km. region and its effects extend to the troposphere and lower stratosphere; (3) when the waves reach a region in which thermal instability is no longer present, they are rapidly damped. The author believes that his theory reconciles the thermal and dynamical theories of cyclogenesis. *G. C. McVittie.*

Kitkin, P. A. On the circulation generated by the wind in a sea of variable density. Izvestiya Akad. Nauk SSSR. Ser. Geofiz. 1953, 232–251 (1953). (Russian)

L'auteur étudie l'influence du vent sur un liquide à densité croissante qui remplit un bassin de forme circulaire. Pour résoudre le problème l'auteur utilise les équations linéairisées du mouvement, en supposant que la composante verticale de l'accélération est nulle. L'auteur forme l'équation d'incompressibilité en supposant qu'en chaque point donné la densité varie seulement par suite du déplacement du liquide à densité variable, tandis que la particule liquide reste invariable. On suppose en outre que la densité peut être représentée sous la forme $\rho(r, z, t)=\rho_0(z)+\rho_1(r, z, t)$ où ρ_0 est la valeur de la densité au repos et ρ_1 est une quantité supplémentaire fonction du rayon, de l'altitude et du temps, et peut être représentée sous la forme: $\rho_1=h(r, t)z^m$. D'après l'auteur ces hypothèses s'accordent assez bien avec les modèles théoriques des perturbations atmosphériques de N. E. Kotchine [voir Trudy Glavn. Geofiz. Observ. 4, 21–45 (1935)].

L'auteur commence à résoudre le cas simple d'un mouvement permanent d'un liquide composé de deux couches: l'une supérieure à densité constante et une autre inférieure à densité croissante avec la profondeur. En introduisant les conditions aux limites et en décomposant les fonctions inconnues: vitesse du courant, l'élévation de la surface libre et l'intensité tangentielle du vent, en séries suivant les fonctions de Bessel, on obtient une solution générale du problème. L'auteur applique ces résultats à un exemple numérique pour un système des vents cycloniques. Par une mèthode analogue, l'auteur étudie le cas d'un mouvement non permanent et applique le résultat au même exemple numérique.

M. Kiveliovitch (Paris)

Kitkin, P. A. Increase of the amplitude of tides in the depths of closed seas. Doklady Akad. Nauk SSSR (N.S.) 91, 237–240 (1953). (Russian)

L'auteur étudie les particularités du mouvement d'un fluide à densité variable dans un bassin cylindrique tournant sous l'influence d'une force massique radiale (f), qui varie avec le temps suivant une loi harmonique. L'auteur montre qu'il est possible de déterminer les composantes de la vitesse, la pression et la densité, si l'on connait d'avance $E=-(1/\rho_0)(d\rho_0/dz)$, où $\rho_0(z)$ est la valeur statique de la densité, et z est l'altitude. En posant $E=\sigma z^m$ on en déduit que l'amplitude des oscillations verticales augmente sous l'influence de la force f dans l'épaisseur de l'eau à densité croissante. On constate, en outre, la formation d'une circulation horizontale, qui compense les grands déplacements verticaux que l'on observe dans les profondeurs intermédiaires.

M. Kiveliovitch (Paris).

Elasticity, Plasticity

***Bezuhov, N. I. Teoriya uprugosti i plastičnosti. [Theory of elasticity and plasticity.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 420 pp. 9.30 rubles.

About three quarters of this book are devoted to the mathematical theory of elasticity, the only unconventional feature being frequent historical notes in which Saint Venant seems to figure as one of the last foreign contributors to the field. The last quarter is an introduction to the theory of plasticity which gives an excellent survey of Russian work on this subject. *W. Prager* (Providence, R. I.).

***Wang, Chi-Teh. Applied elasticity.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. ix+357 pp. $8.00.

This book covers the field with considerable rigour. Assumptions and approximations of course need to be made, but these are clearly stated and discussed at the beginning of each topic. The topics treated in detail are of practical importance. The book can be recommended both for a mathematician and for an engineer.

In the first three chapters stress, strain and stress-strain are treated. Results are proved for two dimensions and stated for three. Tensors are not used. The reviewer feels that a better procedure would have been to write a preliminary chapter on Cartesian tensors. The extra labour required of the student would be more than returned by the ease with which stress and strain can then be treated. Chapter 4 treats plane-strain and plane-stress problems and chapter 5 the torsion problem by inverse and semi-inverse methods.

Chapter 6 is an introduction to finite-difference methods, in particular relaxation. The reviewer thinks that this chapter could be made more concise. Chapters 7 and 8 treat the use of variational methods and of complex variable respectively. Both treatments are good but the use of tensors would shorten the proofs in the former. Chapters 9 and 10 are concerned with the buckling of bars and chapters 11 and 12 with thin plates and shells. The author starts from the displacements in the theory of plates. To the reviewer, this is original and more aesthetically satisfactory than the usual treatment. *D. R. Bland.*

Reissner, Eric. On a variational theorem for finite elastic deformations. J. Math. Physics **32**, 129–135 (1953).

Im Verzerrungstensor werden die quadratischen Grössen nicht vernachlässigt. Für die Beziehungen zwischen den Pseudospannungen—das sind die Spannungen, welche auf das Flächenelement vor der Deformation wirken—und den Verzerrungsgrössen wird angenommen, dass die Verzerrungskomponenten sich als Differentialquotienten einer von diesen Pseudospannungskomponenten abhängigen Funktion W darstellen lassen. Es wird ein Variationsproblem aufgestellt, wobei für einen Teil des Randes die Spannungen, für den anderen Teil des Randes die Verschiebungen als vorgegeben erscheinen, aus dem sowohl die Gleichgewichtsgleichungen, als auch die Beziehungen zwischen Spannungs- und Verzerrungstensor, sowie auch die Randbedingungen folgen. Genauer betrachtet wird der Fall linearer Beziehungen zwischen den Komponenten der Pseudospannungen und der Verzerrungsgrössen. Ferner wird die Funktion W für eine R. S. Rivlin [Philos. Trans. Roy. Soc. London. Ser. A. **240**, 459–490 (1948); diese Rev. **10**, 168] erörterte Beziehung zwischen Spannungs- und Verzerrungsgrössen aufgestellt. Schliesslich wird auch der Fall der endlichen Plattendurchbiegung besprochen. *P. Funk* (Wien).

Adkins, J. E., Green, A. E., and Shield, R. T. Finite plane strain. Philos. Trans. Roy. Soc. London. Ser. A. **246**, 181–213 (1953).

In a straightforward way the authors define finite plane strain, and for the case of plane strain superposed on finite normal extension they derive the appropriate specializations of the general equations of elasticity. Airy's stress function, of course, exists as usual. [The authors are apparently unaware of the work of Finzi, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **19**, 578–584 (1934); and of Sobrero, ibid. **21**, 264–269 (1935).] They then re-derive Rivlin's solution for bending of a rectangular block [by a method which seems to differ from that of the reviewer, J. Rational Mech. Anal. **1**, 125–171, 173–300 (1952), §42I; these Rev. **13**, 794, only in notation]. This solution they are able to generalize to the case when the block is initially a portion of a circular cylindrical shell. Next they construct complex variable formulations of the problem. Taking z as the complex position in the deformed body and D as the complex displacement, they obtain for the incompressible case the fundamental equations

$$\frac{\partial D}{\partial z}+\frac{\partial \bar{D}}{\partial \bar{z}}-\frac{\partial D}{\partial z}\frac{\partial \bar{D}}{\partial \bar{z}}-\frac{\partial D}{\partial \bar{z}}\frac{\partial \bar{D}}{\partial z}=1-\lambda,$$

$$\lambda^2\frac{\partial^2\phi}{\partial z^2}=\frac{\partial \bar{D}}{\partial z}\left(\frac{\partial D}{\partial z}-1\right)H,$$

where λ is the assigned uniform extension normal to the plane, ϕ is Airy's stress function, and H is a certain function of D, $\bar{D}$, and λ determined by the form of the strain energy. For the problem formulated with respect to co-ordinates in the undeformed body they obtain somewhat more complicated equations. They set up formal perturbation schemes for solving both sets, for the case $\lambda=1$. By using these they obtain second-order solutions to the problems of uniform tension of a sheet containing an initially circular hole, an ultimately circular hole, or a rigid circular inclusion. Their resulting value for the hoop stress at the hole is

$$T\left\{(1-2\cos 2\theta)+\frac{3T}{4E}(1\mp 2\cos 2\theta\pm 2\cos 4\theta)\right\},$$

where the upper signs refer to the first problem, the lower to the second. *C. Truesdell* (Bloomington, Ind.).

Platrier, Charles. Conditions d'intégrabilité du tenseur de déformation totale dans une transformation finie d'un milieu à trois dimensions. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 490–494 (1953).

Finite deformation theory in an elastic medium is studied. By direct calculation, necessary conditions satisfied by the strain tensor are obtained. These relations are the desired generalizations of the St. Venant compatability conditions. An alternative method for obtaining these relations is by requiring that the Riemann tensor of the deformed medium vanish [F. D. Murnaghan, Finite deformation of an elastic solid, Wiley, New York, 1951, pp. 38–41; these Rev. **13**, 600; I. S. Sokolnikoff, Tensor analysis, ibid., 1951, p. 300; these Rev. **13**, 584]. Since the vanishing of the Riemann tensor is a sufficient condition that the deformed medium be Euclidean [T. Y. Thomas, The differential invariants of generalized spaces, Cambridge, 1934, p. 216; I. S. Sokolnikoff, op. cit., pp. 96–100], these conditions are also sufficient that the strain tensor determine a mapping of a given Euclidean space into a Euclidean space and hence a deformation vector. [See also C. Truesdell, J. Rational Mech.

Anal. 1, 125–171, 173–300 (1952), p. 145; these Rev. 13, 794.] *N. Coburn* (Ann Arbor, Mich.).

Sternberg, E. On Saint-Venant's principle. Quart. Appl. Math. 11, 393–402 (1954).

The author has made an important contribution to the linear theory of elasticity by establishing the validity of Saint-Venant's principle, as modified by von Mises. Let D be a region in space with boundary B. Let $P_0^{(k)}$ $(k=1, \cdots, m)$ denote m distinct points on B and, for each $\epsilon>0$, let $S_k(\epsilon)$ $(k=1, \cdots, m)$ denote m disjoint subregions of B such that $S_k(\epsilon)$ is contained in the sphere of radius ϵ with center at $P_0^{(k)}$. Let $e_{ij}(\epsilon)$ denote the strain components constituting a solution to the equations of linear elasticity for isotropic materials such that, on B, the stress vector $\mathbf{T}(\epsilon)=[X_1(\epsilon), X_2(\epsilon), X_3(\epsilon)]$ vanishes except on the subregions $S_k(\epsilon)$. One can express $e_{ij}(\epsilon)$ as the sum of m tensors $e_{ij}[S_k]$ such that $e_{ij}[S_k]$ represents, in a formal way, the contribution to $e_{ij}(\epsilon)$ from the tractions on $S_k(\epsilon)$. Let $\mathbf{F}_k(\epsilon)$ and $\mathbf{M}_k(\epsilon)$ denote the resultant force and resultant moment in $P_0^{(k)}$ of the tractions on $S_k(\epsilon)$. The author shows that, subject to certain conditions of regularity, if $\mathbf{T}(\epsilon)$ is bounded as $\epsilon\to 0$ on $S_k(\epsilon)$, then, at a fixed interior point of D, as $\epsilon\to 0$,

$$e_{ij}[S_k]=O(\epsilon^3) \text{ or smaller if } \mathbf{F}_k(\epsilon)\neq 0,$$

$$e_{ij}[S_k]=O(\epsilon^3) \text{ or smaller if } \mathbf{F}_k(\epsilon)=0,$$

$$e_{ij}[S_k]=O(\epsilon^4) \text{ or smaller if } \int_{S_k(\epsilon)} X_i(\epsilon)\,d\sigma=0$$

$$\text{and } \int_{S_k(\epsilon)} X_i(\epsilon)x_j\,d\sigma=0.$$

The latter conditions imply that $\mathbf{F}_k(\epsilon)=\mathbf{M}_k(\epsilon)=0$, but the converse is not true, in general.

Define a concentrated force at a point of B as the limit of a sequence of resultant forces over areas of B which converge to this point. Let the concentrated forces $\mathbf{T}_n(\epsilon)$ $(n=1, \cdots, N)$ be applied at points $A_n^k(\epsilon)$ of $S_k(\epsilon)$ and assume these forces are bounded as $\epsilon\to 0$. Then the results stated above hold, provided the integrals are replaced by the corresponding finite sums and $O(\epsilon^2)$, $O(\epsilon^3)$, $O(\epsilon^4)$ are replaced by $O(1)$, $O(\epsilon)$, $O(\epsilon^2)$, respectively.

J. L. Ericksen (Washington, D. C.).

Washizu, K. Bounds for solutions of boundary value problems in elasticity. J. Math. Physics 32, 117–128 (1953).

This paper considers the problem of obtaining bounds for the solutions of the two- and three-dimensional stress problems, the bending of thin plates and torsion in bars. The method is predominantly that of Diaz and Greenberg [same J. 27, 193–201 (1948); these Rev. 10, 213] since the author uses their method in the definition of the Green's functions. However, the geometrical representation in function-space introduced by Prager and Synge [Quart. Appl. Math. 5, 241–269 (1947); these Rev. 10, 81] and used in the Maple-Synge [e.g., Maple, ibid. 8, 213–228 (1950); these Rev. 12, 704; Synge, Proc. Roy. Soc. London. Ser. A. 208, 170–175 (1951); these Rev. 13, 235] method is utilized to obtain some of the fundamental inequalities and for improving the accuracy. Examples are given to illustrate the methods for each of the previously mentioned problems.

C. G. Maple (Ames, Iowa).

Filonenko-Borodič, M. M. Some generalizations of Lamé's problem for an elastic parallelepiped. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 465–469 (1953). (Russian)

Earlier papers [same journal 15, 137–148, 563–574 (1951); these Rev. 13, 92, 794] are concerned with a rectilinear parallelepiped. In this paper the author finds series expansions, in cylindrical coordinates, for three stress functions which give zero stress on the whole surface of a finite cylinder whose cross-section is a concentric circular annulus or a sector of the annulus. *L. M. Milne-Thomson.*

***Muskhelishvili, N. I. Some basic problems of the mathematical theory of elasticity. Fundamental equations, plane theory of elasticity, torsion and bending.** Translated by J. R. M. Radok. P. Noordhoff, Ltd., Groningen, 1953. xxxi+704 pp. 38.00 Dutch florins.

A translation of the 3rd edition of Nekotorye osnovnye zadači matematičeskoĭ teorii uprugosti [Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1949; these Rev. 11, 626]. In the bibliography names of Russian authors are given in the cyrillic alphabet (but unfortunately with many misprints) as well as in a transliterated version in the latin alphabet.

***Savin, G. N. Koncentraciya napryaženiĭ okolo otverstiĭ. [Concentration of stresses around openings.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 496 pp. 17.50 rubles.

This book contains a rounded account of the current state of the problem of stress concentration in the neighborhood of holes in thin elastic plates. It is intended primarily for the practicing stress analyst, and, for this reason, presents many final results either as simple formulas or in graphical and tabular forms. The major part of the work is based on the application of the techniques developed by Mushelišvili for the solution of problems in plane elasticity. The book consists of eight chapters, which, with the exception of the first, can be read independently of the others.

Chapter 1, pp. 1–55, summarizes the Mushelišvili method. The presentation here is so condensed that the reader not already familiar with the method will find it necessary to consult the original monograph by Mushelišvili [Some basic problems of the mathematical theory of elasticity, 3rd ed., Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1949; these Rev. 11, 626; an English translation is reviewed above]. This chapter appears to have been hurriedly written and it is marred by a few misstatements and inaccuracies.

Chapter 2, pp. 56–186, is given to an analysis of stress near openings in plane elastic sheets subjected to various types of loading. A detailed discussion of the stress concentration near the circular elliptical, triangular and square holes with rounded corners, is included. Some results are also given for the stress concentration near a circular hole in a circular cylindrical shell, and in a semi-infinite sheet and in an infinite sheet with several circular holes.

Chapter 3, pp. 185–242, is concerned with the stress distribution near an elliptical hole in a thin anisotropic plate. Chapter 4, pp. 243–276, deals with several problems on stress concentration in an elastic-plastic deformation. Chapter 5, pp. 277–347, contains a discussion of the stress concentration in thin plates with holes reinforced by elastic rings. Chapter 6, pp. 348–406, is devoted to the analysis of the stress concentration in thin elastic plates deflected by loads applied to the boundaries of the plates. Infinite plates with circular, elliptical, and rectangular holes with rounded corners are considered. Chapter 7, pp. 407–452, treats

several cases of the problem of transverse deflection of a thin elastic plate weakened by a reinforced circular hole. Chapter 8, pp. 453–487, summarizes the experimental techniques (principally photo-elastic) used in the study of the stress concentrations, and includes a comparison of the theoretical with experimental results. The book contains a comprehensive bibliography of some 140 items, with the latest entries dated 1950.

Among minor blemishes noted by the reviewer are the following: The characterization of the system of equations (l. 12) on p. 13 is incomplete; eqn. (l. 46) on p. 30 should be supplemented by a statement that two more elastic constants vanish; most of the material on pp. 40–55 can be omitted because it is not needed in the sequel; the symbol Re in the second of eqns. (4.52) on p. 256 should be deleted; the statement on p. 396 concerning the number of vanishing elastic constants for orthotropic materials is incomplete, since c_{45} is also zero. None of these invalidates the results of this most useful book. *I. S. Sokolnikoff.*

Šerman, D. I. On stresses in a plane heavy medium with two identical symmetrically placed circular openings. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 751–761 (1951). (Russian)

A homogeneous and isotropic elastic material fills the semi-infinite triply-connected domain S, bounded by the straight line L_0 parallel to the X-axis, and by two non-intersecting circles L_1 and L_2 with equal radii R. The centers of the circles lie on the X-axis at a distance f from L_0. The material filling S is acted on by a uniform gravitational force in the direction of the Y-axis, and the boundaries L_0, L_1, L_2 are free of external loads. A solution of this two-dimensional elastostatic problem, in the neighborhood of L_1 and L_2, is obtained under the hypothesis that $R \ll f$. The author utilizes the Muskhelišvili formulations of such problems.

I. S. Sokolnikoff (Los Angeles, Calif.).

Šeremet'ev, M. P. The plane stressed state of a plate with a reinforced circular opening. Akad. Nauk SSSR. Inženernyĭ Sbornik **14**, 81–100 (1953). (Russian)

An opening in an infinite elastic plate is reinforced by a ring which is welded onto the plate along the contour C of the opening. The reinforcing ring is assumed to act as a curved rod offering resistance to extension and bending and the plate is supposed to be in the state of plane stress under suitably distributed loads on its boundaries. The state of stress in the plate is then determined by two analytic functions φ_i of a complex variable. The elastic properties of the ring and the equilibrium equations of the thin-rod theory serve to determine the boundary conditions satisfied by the φ_i along C. The resulting boundary-value problems for the determination of the φ_i is one of the standard problems treated by Muskhelišvili. The author deduces the boundary conditions and determines the state of stress in a plate reinforced by a circular ring when the plate is subjected to bi-axial tensions at infinity. The external forces applied to the reinforcing ring are represented by functions of bounded variation. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Narodeckiĭ, M. Z. Stretching of a square plate weakened by a circular cut-out in the center. Akad. Nauk SSSR. Inženernyĭ Sbornik **14**, 101–108 (1953). (Russian)

Stein, Manuel, and Mayers, J. A small-deflection theory for curved sandwich plates. NACA Rep. no. **1008**, 6 pp. (1951).

This is a reprint of paper published as NACA Tech. Note no. **2017** (1950); these Rev. **11**, 627.

Buzin, E. I. On equilibrium of a flexible extensible surface of positive Gaussian curvature. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **9**, 124–136 (1952). (Russian)

Starting from the stress equilibrium equations, in terms of Gaussian coordinates, for a flexible inextensible material, the author goes over to isotropic slightly extensible material by a small variation in which the external forces are kept unchanged. This with the aid of Hooke's law leads to a system of equations for the stresses. The scope is that of the linear theory. *L. M. Milne-Thomson* (Greenwich).

Šapiro, G. S. Some problems on deformations of rods of variable cross-section. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 249–252 (1953). (Russian)

In this paper the deformations of two kinds of rods are considered: first, a rotationally symmetric rod with variable radius of the circular cross-section, loaded by thrust on the end-planes and arbitrary (rotational-symmetric) forces on the curved surfaces; second, a rectangular rod with one dimension of the rectangular cross-section changing along the span of the rod, loaded by thrust on the end-planes. In the first case the stresses are derived from one stress-function; in the second case two stress-functions are needed. The stress-functions are split up into a part depending on the boundary conditions and a part equal to zero on the boundaries parallel to the axis of the rod. The first part is calculated directly from the boundary conditions and the second part can be found by some variational method (Ritz, Galerkin). *W. H. Muller* (Amsterdam).

Ōkubo, H. The torsion and stretching of spiral rods. II. Quart. Appl. Math. **11**, 488–495 (1954).

Guided by the results of an earlier paper [same Quart. **9**, 263–272 (1951); these Rev. **13**, 91] the author chooses suitable forms for the displacement components, which degenerate to known tension and torsion systems by properly limiting the various parameters. The equilibrium equations for displacements are satisfied up to three arbitrary analytic functions of the coordinates for the plane normal to the axis of the helix. These in turn are determined by the boundary requirement on the lateral surface of the rod. In the example of an elliptical section of a helix, a conformal mapping of this section onto a unit circle is made and suitable analytic functions provide the necessary solution. The special case of circular section is treated and graphs are shown for the peripheral shearing stress and axial stress.

D. L. Holl (Ames, Iowa).

Wilhoit, J. C., Jr. An addition to Poritsky's solutions of a differential equation of torsion. Quart. Appl. Math. **11**, 499–501 (1954).

Poritsky [Proc. Symposia Appl. Math., v. 3, pp. 163–186, McGraw-Hill, New York, 1950; these Rev. **12**, 458] determines the torsion stress function $\psi(r, z)$ in cylindrical coordinates for axially symmetric shafts of revolution. In particular, ψ may be expressed in polar spherical coordinates by products of powers of R and Legendre functions of the first and second kind, with two missing solutions, one containing R, and another R^2. The author differs from Poritsky

on these missing solutions; however, he verifies them as satisfying the required equations. Two additional solutions are obtained in spherical coordinates for relaxed boundary conditions that ψ have a derivative in R behaving as $1/R$ on the boundary. *D. L. Holl* (Ames, Iowa).

Šerman, D. I. Torsion of an elliptic cylinder stiffened with a circular rod. Akad. Nauk SSSR. Inženernyĭ Sbornik **10**, 81–108 (1951). (Russian)

A detailed solution of Saint-Venant's torsion problem is given for a homogeneous and isotropic elliptical beam reinforced by a circular rod. The rod is welded onto the cylinder along the lateral surface, and the axes of the rod and beam coincide. This is a generalization of the corresponding torsion problem for an elliptical beam weakened by a cylindrical cavity, solved by D. I. Šerman and M. Z. Narodeckiĭ [same Sbornik **6**, 17–46 (1950); these Rev. **13**, 886]. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Basilewitsch, W. Das Torsionsproblem der ⊤, ⊏ und ⌝ Träger. Acad. Serbe Sci. Publ. Inst. Math. **5**, 5–20 (1953).

The torsion problem for bars whose sections are rectangles arranged to form a ⊤, ⌝ or ⊏ section is accomplished by choosing two Prandtl torsion functions $\phi_1(x, y)$ and $\phi_2(x, y)$, each satisfying $\nabla^2\phi_i = -2$ in the two rectangular subregions. Each ϕ_i vanishes on two straight-line boundary segments of its subregion and is made to vanish along a third boundary by requiring that there exist one relationship among its infinite set of constants. The remaining boundary of each rectangle is partly or wholly the common boundary of the joined subregions. To satisfy the continuity of ϕ_i and their normal derivatives along this common junction as well as the boundary condition on the extension of the line of junction, it is necessary to get Fourier expansions for special factors of each ϕ_i. This leads to the determination of the constants by solving an infinite linear system of infinitely many unknowns. Although no tests are shown, it appears that the convergence permits one to solve an upper left hand segment of this system for a finite number of the unknowns. Numerical examples for ⊤, ⌝ and ⊏ sections are completed; torsion moments calculated and shear lines shown for each section. *D. L. Holl* (Ames, Iowa).

Basilevich, V. Shearing stress in bending of I beams. Acad. Serbe Sci. Publ. Inst. Math. **5**, 21–28 (1953).

The author solves the flexure problem of an I section by choosing two harmonic flexure functions ϕ_1, ϕ_2 for each of two rectangular subregions and satisfying the required boundary condition of their normal derivatives on three sides of their respective rectangular regions. Along their common junction the method described above (see preceding abstract) leads to a determination of all the constants in each flexure function after solving one linear infinite system. A numerical example is given and results are obtained by solving a segment of five equations. From these, stresses are obtained which are said to be confirmed by the soap film method. *D. L. Holl* (Ames, Iowa).

Langhaar, H. L. On torsional-flexural buckling of columns. J. Franklin Inst. **255**, 101–112 (1953).

A stationary conservative mechanical system is in a state of stable equilibrium only if its potential energy V has a relative minimum; that is, $\Delta V > 0$ for any small virtual displacement of the system that is consistent with the constraints. For an elastic system, $V = U + \Omega$, where U is the strain energy and Ω is the potential energy of the external forces; consequently, $\Delta V = \Delta U + \Delta\Omega$. If the system has linear elasticity, $\Delta U = \sum_{n=1}\delta^n U/n!$ where $\delta^n U$ (the nth variation of U) is the volume integral of a homogeneous polynomial of nth degree in the components of the virtual displacement vector and its first derivatives. In many important cases, Ω depends linearly on the virtual displacements. Then $\Delta\Omega = \delta\Omega$ and $\Delta V = \delta\Omega + \sum_{n=1}\delta^n U/n!$. For any equilibrium configuration, $\delta\Omega + \delta U = 0$. Consequently, since $\delta^2 U$ depends on all components of the virtual displacement vector, it has the same sign as ΔV, provided that the virtual displacements are small. Therefore, when $\delta^2\Omega$ vanishes identically, a necessary and sufficient condition for stability of equilibrium is that $\delta^2 U$ be positive definite, that is, $\delta^2 U > 0$ for all virtual displacements that are consistent with the constraints. When the type of loading on an elastic structure is specified, the external loads are determined completely by the magnitude P of the total load. For small values of P, the second variation $\delta^2 U$ is positive definite, but for large values of P, $\delta^2 U$ is indefinite. The buckling load P_{crit} is the value of P, for which $\delta^2 U$ experiences this change of character. If (ξ, η, ζ) is the displacement vector that defines the given equilibrium configuration, (u, v, w) the incremental virtual displacement vector, and if rectangular coordinates (x, y, z) are used, the exact expressions for the strain components ϵ_x and γ_{yz} are

$$\begin{aligned}\epsilon_x &= \xi_x + u_x + \tfrac{1}{2}(\xi_x + u_x)^2 + \tfrac{1}{2}(\eta_x + v_x)^2 + \tfrac{1}{2}(\zeta_x + w_x)^2 + \cdots,\\ \gamma_{yz} &= \eta_z + v_z + \zeta_y + w_y + (\xi_y + u_y)(\xi_z + v_z)\\ &\quad + (\eta_y + v_y)(\eta_z + v_z) + (\zeta_y + w_y)(\zeta_z + w_z),\end{aligned} \tag{1}$$

with similar expressions for ϵ_y, ϵ_z, γ_{zx} and γ_{xy}. The strain energy of the system is

$$U + \Delta U = \tfrac{1}{2}G\int\int\int\left\{\frac{2\nu}{1-2\nu}(\epsilon_x+\epsilon_y+\epsilon_z)^2 + 2(\epsilon_x^2+\epsilon_y^2+\epsilon_z^2) + (\gamma_{yz}^2+\gamma_{zx}^2+\gamma_{xy}^2)\right\}dxdyds. \tag{2}$$

C. Lanczos pointed out in an unpublished paper (1943) that when the quadratic terms in eq. (1) (and corresponding terms in the expressions for ϵ_y and ϵ_z) are neglected, $\delta^2 U$ is invariably positive definite. Consequently, in cases for which $\delta^2\Omega$ vanishes identically, the phenomenon of buckling does not lie within the scope of the linear theory of elasticity. This conclusion might have been anticipated from the uniqueness theorem of linear elasticity theory. The distinguishing feature of the given treatment of torsional-flexural buckling of columns is that it does not adduce a theory of non-uniform torsion. The equations are derived analytically from the fundamental equationss of elasticity and the variational principles of mechanics. Agreement with the theory of R. Kappus [Luftfahrtforschung **14**, 444–457 (1937) = NACA Tech. Memo. no. **851** (1938)] is not obtained, unless a term that is inversely proportional to the square of the length of the column, can be neglected. The effect of this approximation is illustrated by an example. *R. Gran Olsson* (Trondheim).

Bondar', N. G. Dynamical stability and vibrations of hingeless parabolic arcs. Akad. Nauk SSSR. Inženernyĭ Sbornik **13**, 87–102 (1952). (Russian)

This paper deals with bending vibrations of a symmetrical parabolic arc, clamped at both ends, and takes into consideration that the shape of the arc changes during the vibration. By this, the problem becomes non-linear. First the forced vibrations of a massless elastic arc with a weight

in the midpoint (apex) under a periodic force in the same point are dealt with. The equation of motion becomes a very complicated non-linear integro-differential equation. An approximate solution of this equation is found by substituting for the term representing the inertia-forces the expression calculated from the corresponding linear problem, i.e. the same problem solved by neglecting the change of the shape of the arc. Then the mode of vibration is determined by Mathieu's equation, from which can be concluded what choices of the weight in the midpoint, the amplitude and the frequency of the applied force will lead to instability.

In the second part of the paper the vibrations of a homogeneous heavy arc under a uniformly distributed periodic load are treated in the same way. *W. H. Muller.*

Gröbner, W., und Lesky P. Eigenschwingungen eines Kreisringes mit rechteckigem Querschnitt. Österreich. Ing.-Arch. **7**, 254–262 (1953).

The authors determine approximately the characteristic frequencies for the free elastic vibrations of a ring of rectangular cross-section. An application of the Ritz method enables the authors to reduce the solution of the problem to a system of ordinary differential equations for the time-dependent coefficients of the expansions for displacements. Only the case of plane strain is discussed. No comparison is given with the previous exact solution [Federhofer, Akad. Wiss. Wien, S.-B. IIa. **144**, 561–575 (1935)]. *E. T. Onat* (Providence, R. I.).

Cole, J. D., Dougherty, C. B., and Huth, J. H. Constant-strain waves in strings. J. Appl. Mech. **20**, 519–522 (1953).

The authors obtain a solution containing a velocity wave to the equations of motion for an elastic string in which the tension is assumed to be a linear function of the change in length per unit initial length. *J. L. Ericksen.*

Satô, Yasuo. Velocity of elastic waves propagated in media with small obstacles. Bull. Earthquake Res. Inst. Tokyo **31**, 1–18 (1953). (Japanese summary)

As in the previous paper [same Bull. **30**, 179–190 (1952); these Rev. **14**, 1147], the author assumes that the obstacles in a medium are spherical bodies. In this paper solid obstacles are considered instead of holes filled with a liquid, but under similar conditions (large number of obstaclel located in a small part of the whole volume). Using exs pressions for displacements and stresses in terms of spherica-harmonics and an "equivalent" homogeneous medium, the unknown coefficients are found from the boundary conditions. Two equations, for the bulk modulus and the rigidity, are derived and some examples of variations of elastic constants and velocities of compressional and shear waves are given. These variations depend on the proportion of volume occupied by obstacles. *W. S. Jardetsky.*

***Bullen, K. E. An introduction to the theory of seismology.** 2d ed. Cambridge, at the University Press, 1953. xv+296 pp. $6.50.

Although "the general pattern of the first edition [1947; these Rev. **10**, 88] has been preserved", a twenty-page bibliography has been added and various minor changes have been made.

Kačanov, L. M. Stability of elastic-plastic equilibrium of a compressed twisted shaft. Doklady Akad. Nauk SSSR (N.S.) **88**, 627–630 (1953). (Russian)

The stability of a shaft of uniform, circular cross-section, subjected to the simultaneous action of twisting couple M and axial thrust P, is studied for an incompressible and non-strain-hardening material obeying the Mises yield condition. The author chooses to use a stress-strain relation of the total-strain (Hencky) type for describing the stress state in the straight shaft, and a stress-strain relation of the incremental (Prandtl-Reuss) type for determining the stress increments in bending. The latter relation is attributed to Shepherd [Inst. Mech. Engrs. Proc. **159**, 95–114 (1948); these Rev. **10**, 648], but no reference is given for the former, [which is apparently that given by Sokolovsky, Theory of plasticity, Izdat. Akad. Nauk, Moscow-Leningrad, 1946; these Rev. **8**, 545]. It is then shown that this problem reduces to that for the elastic shaft [Greenhill, Inst. Mech. Engrs. Proc. **1883**, 182–225] if the elastic flexural rigidity for the latter is replaced by an appropriate "plastic" flexural rigidity called A. Here A is determined for some special cases, and a simple formula for A is given for a circular thin-walled tube. No numerical examples are included.

In this formulation, A is found to be independent of the manner in which the twist and compression are applied to the shaft and so is determined in each case as a single-valued function of P and M. The author does not mention that the stress distribution as given by the total-strain law is at best only a fair approximation to that for a Prandtl-Reuss material, in which, for combined tension and torsion, the distribution of stress for a given state of deformation may depend upon the path of loading [see Prager and Hodge, Theory of perfectly plastic solids, Wiley, 1951, Section 14; these Rev. **14**, 430]. If incremental theory were used, A could not then depend only upon P and M. However, this theory may have useful engineering applications. *W. Nachbar* (Seattle, Wash.).

Wang, Alexander J. Plastic flow in a deeply notched bar with semi-circular root. Quart. Appl. Math. **11**, 427–438 (1954).

The theory of the slip-line field in plane plastic flow is used to investigate the unsteady-motion problem of a circular-notched bar pulled in tension. The basic equations governing stress and velocity fields are integrated in the characteristic plane for the case of a circular stress-free boundary. The results of this integration are used to construct the boundary change in a step-by-step manner, since at each step the new free boundary of the plastic region can be approximated by a circle. The final shape of the boundary of an initially semi-circular notch is obtained when plastic flow has reduced the initial connection at the root to a line contact between the shanks. *E. T. Onat.*

MATHEMATICAL PHYSICS

Weyssenhoff, Jan. On the microstructure of the world. I. The elementary length. Acta Phys. Polonica **11**, 273–297 (1953).

There has long been the suspicion among physicists that the Minkowski geometry of space-time—the study of configurations of point-events which are invariant under the 10-parameter Lorentz group—is ill-adapted to the requirements of quantum physics. In furtherance of this viewpoint, the author argues that the point (or point-event) is suitable only for macrophysics, and that for microphysics it must

be replaced by some more appropriate primitive element. This element he finds in the notion of a "directed wave front", and as the primitive relation of the geometry the "concordant contact" between two such wave fronts. The resulting physical geometry is then that based on the 15-parameter group of Lie's transformations of 3-dimensional spheres, which is isomorphic to the Möbius group of conformal transformations in four dimensions. This group, under which Maxwell's equations for free space are invariant, has been employed in optics by Bateman and by Cunningham, but has never been fully exploited in particle physics. The principal concrete result of the present paper is that at "large distances"—the macroscopic domain—the Lie group approaches the group of Euclidean motions plus dilatations. The precise formulation of this result involves the introduction of a fundamental length l; it is suggested that the more complete theory will involve the introduction of the two additional fundamental constants c and h, an aspect to be dealt with in subsequent papers of the series.

H. P. Robertson (Pasadena, Calif.).

Optics, Electromagnetic Theory

Weidert, Fr. Über die Berechnung des Anteils der einzelnen Flächen an der Gesamtaberration eines optischen Systems. Optik **10**, 375–388 (1953).

To evaluate the contribution of a certain surface, say the 4th, to an image aberration, it is suggested that one consider the aberration obtained by tracing rays up to this surface by exact calculation and through the rest of the system by a paraxial calculation. If the same type of aberration is then calculated relative to the $(K-1)$st surface and the results subtracted, an estimate is obtained for the effort of the Kth surface on this aberration. Consideration is given to the previous formulae given in the literature relative to this problem, and an actual calculation is given to show how other estimates differ from those in the present paper for the contributions of the individual surfaces to the composite image aberrations. *M. Herzberger* (Rochester, N. Y.).

Montel, Marc. Classification des aberrations géométriques des systèmes optiques sans symétrie de révolution. Rev. Optique **32**, 585–600 (1953).

By use of the characteristic function, image errors are defined for an unsymmetrical optical system. In doing so a "pseudo-axis" is defined as the broken-line path of the ray intersecting the diaphragm perpendicularly at its center and refracted at the surfaces of the system. The author tries to give a classification of the image errors of such a system and studies the effect of a change of the position of the aperture diaphragm on these errors. Consideration is given to the case of a system with plane or rotational symmetry, and the resulting theory is applied to the investigation of centering errors in rotation-symmetric systems.

E. W. Marchand (Rochester, N. Y.).

Toraldo di Francia, G. On the image sharpness in the central field of a system presenting third- and fifth-order spherical aberration. J. Opt. Soc. Amer. **43**, 827–835 (1953).

The author calculates the light distribution in the image of a sharp edge under the assumption that the image-formation is sufficiently well characterized by an approximation up to the fifth order. The calculation is carried out for different positions of the image plane and the author not only gives contrast curves, but also plots the axial interval along which a certain percentage change in illumination occurs. The curves are thought of as raw material to which each designer may apply his favorite criterion and draw his own conclusions. The computation is based on geometrical optics, and the author states that, for large wave aberrations, the diffraction calculations would not substantially influence the results. *M. Herzberger* (Rochester, N. Y.).

Meinel, A. B. Aspheric field correctors for large telescopes. Astrophys. J. **118**, 335–344 (1953).

Since the useful field of a large paraboloid is severely limited by coma and spherical aberration, a so-called Ross corrector system has been introduced. This corrector, which is constructed with spherical surfaces, increases the useful field of the 200-inch Hale telescope, for example, to a diameter of $\frac{1}{4}°$. But beyond that point the image qualities deteriorate rapidly. The present paper describes a corrector constructed with aspheric surfaces and designed to enlarge the field considerably. Preliminary estimates indicate that a diameter of 1° can be achieved by this means for the case of the 200-inch telescope for monochromatic light.

M. Herzberger (Rochester, N. Y.).

Berezina, L. Ya. Investigation of the passage of a general congruence of rays through an optical system by means of the method of the moving trihedron. Latvijas PSR Zinātņu Akad. Vēstis **1952**, no. 3 (56), 115–126 (1952). (Russian. Latvian summary)

This paper is written for physicists with an ordinary college training in mathematics and interested in geometrical optics. It gives a simple exposition of the elementary theory of rectilinear congruences in the ω-notation; stress is laid on the transformation of one trihedron into another. Consideration of a congruence in a medium of refraction index n_1 into a medium of refraction index n_2 leads to the theorem of Malus and to formulas such as

$$\frac{n_2\cos^2 i'}{r_\alpha}-\frac{n_1\cos^2 i}{r_\alpha}=\frac{n_2\cos i'-n_1\cos i}{R_\sigma},$$

$$\frac{n_2}{r_\beta}-\frac{n_1}{r_\beta}=\frac{n_2\cos i'-n_1\cos i'}{R_\tau}.$$

Here i, i' are the angles which the incoming and the refracted rays r, r' make with the normal to the separation surface (ξ); R_σ is the radius of curvature of that normal section of (ξ) in the osculating plane of which lie r, r'; R_τ the radius of curvature of that normal section of (ξ) orthogonal to the first; r_α, r_α are the radii of curvature of the normal sections which the plane through r and r' carves into the incoming and refracted wave surfaces; r_β, r_β the radii of curvature of the normal sections orthogonal to these. The paper finishes with some remarks on homocentric pencils and on multiple refractions. *D. J. Struik.*

Glaser, W. Über die Bewegung eines "Wellenpakets" in einer Elektronenlinse. Österreich. Ing.-Arch. **7**, 144–152 (1953).

In this paper the author shows that the action of an electromagnetic lens on an electron beam is to limit its spreading as it passes through it. This makes it possible to get an image of a real object by employing electron lenses. The mathematical proof of this statement is based on the

solution of the paraxial wave equation which is given in the first part of the paper and is similar to that of the paper reviewed below. The special case of an elliptic object which has an electron density distribution in a gaussian form is analyzed in detail. Due to the lens action of the electromagnetic field, the current density on the image plane is found to have the same shape (elliptic) as that of the object, although the amplitude depends on the axial coordinate. The variation of the cross-section of the original electron beam is interpreted as being due to the interaction of the classical motion of the electrons with the field and the principle of uncertainty. *N. Chako* (New York, N. Y.).

Glaser, Walter, und Schiske, Peter. Elektronenoptische Abbildung auf Grund der Wellenmechanik. I. Ann. Physik (6) **12**, 240–266 (1953).

Starting from the Schrödinger equation of an electron in an external electromagnetic field of rotational symmetry, the authors have derived the paraxial wave equation (p.w.e.) of electron optics. This is based on the assumption that the moments in the x and y directions are small in comparison to that in the axial direction. The resulting p.w.e. is two-dimensional and of parabolic type. An alternative method based on Hamilton's principle has also been given. By expressing the amplitude wave function in Debye's form, (1) $\psi(x, y, z)=a(z)\exp[ik^{-1}S(x, y, z)]$, the authors obtain the general solution of the p.w.e. in an integral representation (Whittaker's type) containing an arbitrary function in the variables of integration, which afterwards is identified, up to constant quantities, as the limiting value which the solution itself takes on the surface (plane) of integration.

Since the expansion of the phase function $S(x, y, z)$ is limited to linear and square terms in x, y, with the coefficients being functions of z, the integral representation is of the same form as Kirchhoff's formulation for the Fresnel diffraction phenomena. It is also shown that the current distribution on an image plane varies inversely as the square of one of the particular solutions of the paraxial ray equation, i.e., $s(z_1)\neq 0$. In the next section the authors treat the problem of diffraction by a square and a circular aperture. The results which are obtained under the above mentioned limitations are similar to the corresponding formulae in wave optics, which for the circular aperture are expressed by the two Lommel functions u and v. The arguments of the functions representing the diffracted amplitude depend on the two particular solutions of the paraxial ray equation. In the following section the effect of the limitation of the aperture (finite) on the image wave function is discussed.

Finally the important problem of imagery of periodic objects (grating) is analysed. The expressions for the diffracted amplitude are of similar form as Rayleigh's results for the case of diffraction by a grating. *N. Chako.*

Glaser, Walter, und Schiske, Peter. Elektronenoptische Abbildungen auf Grund der Wellenmechanik. II. Ann. Physik (6) **12**, 267–280 (1953).

In this article the authors have formulated the diffraction theory of aberrations of electron optics. Starting with Schroedinger equation of an electron in external fields, they have derived an integral representation of the amplitude electron wave function similar to the Kirchhoff's formulation in wave optics, by introducing a Green's function, corresponding to the present problem,

$$(1)\qquad \psi(P_1)=\int_{\Sigma_0}\psi(P_0)(\partial G/\partial z)dS \quad (dS=dx_0dy_0).$$

The Green's function G must satisfy the following conditions: (a) it should be continuous and finite to the right of the plane Σ_0 except at the point P_1 where it becomes infinite as the reciprocal of the distance from P_1; (b) it vanishes on Σ_0; (c) it behaves as a spherical wave at infinity. Condition (c) is fulfilled provided that the vector potential vanishes at infinity as $1/R$ and, the scalar potential tends to a constant value. If the singular part G_s of $G=G_s+G_r$ (G_r is regular everywhere) is expressed by $G_s=a(z)\exp[ikR^{-1}S_s(x, y, z)]$, where a is a slow varying function of position, then S_s represents the characteristic function of the corresponding geometrical optics problem. The amplitude a is determined on the assumption that G_s satisfies condition (a) and that the curvature of the path PP_1, is neglected. The regular part G_r is obtained from condition (c) and the corresponding characteristic function $S_r(x, y, z)$ which satisfies a similar Hamilton equation as S_s on Σ_0. The final result is given by the formula

$$(2)\qquad \psi(P)=\frac{1}{2\pi ik}\int_{\Sigma_0}\psi(P_0)\left(\frac{eAz_0-\partial S/\partial z_0}{eAz_0+\partial S/\partial z_0}\right)^{\frac{1}{2}}\Delta e^{ikS(P,P_0)/k}dx_0dy_0$$

which reduces to Kirchhoff's formula for free-field regions. The quantity is a determinant in $(S_{xx_0}\cdots S_{yy_0})$. Finally (2) is applied to an electron optical system of rotational symmetry, by limiting the case to third-order aberrations. The problem is treated from the point of view of the Seidel theory of aberrations, where s is taken as the Seidel eiconal of geometrical electron optics [see further Glaser, Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **159**, 297–360 (1950); these Rev. **14**, 931]. *N. Chako.*

Karp, Sam. Diffraction by a tipped wedge. With applications to blunt edges. Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. **EM-52**. i+18 pp. (1953).

This paper deals with diffraction of electromagnetic waves by a perfectly conducting infinite wedge to which has been fixed a perfectly conducting circular cylinder with axis the edge of the wedge; thus the wedge, instead of having a sharp edge, has a cylindrical tip. The problem is two-dimensional. The incident waves are assumed to be monochromatic and may be either plane waves or waves from an infinitely long line source along a line parallel to the edge of the wedge. If there is no tip, the diffraction problem can be solved by a series of eigenfunctions in cylindrical coordinates. The presence of the tip perturbs this solution, and the additional field is the sum of expanding cylindrical waves which satisfy the boundary conditions on the wedge; the additional field is then uniquely determined by the boundary conditions on the cylindrical tip. The asymptotic behavior of the scattered field at a great distance is found in the case when the incident waves are plane, and graphs of the net scattered amplitude are given for the special case of normal incidence on a half plane with a cylindrical tip of $\lambda/2\pi$ where λ is the wave length. *E. T. Copson* (St. Andrews).

Kalitzin, Nikola St. Eine Verallgemeinerung der Gleichungen der Elektrodynamik. C. R. Acad. Bulgare Sci. **4** (1951), no. 2–3, 17–20 (1953). (Russian summary)

Contains part of the material given in Kalitzin, Izvestiya Bulgar. Akad. Nauk. Otd. Fiz.-Mat. Tehn. Nauk. Ser. Fiz. **2**, 63–78 (1952); these Rev. **15**, 169. *N. Rosen.*

Arsove, Maynard G. A note on the network postulates. J. Math. Physics **32**, 203–206 (1953).

A new postulational approach is made to the basic network formula $I=A(A^*ZA)^{-1}A^*E$ by retaining Kirchhoff's first (node) law but replacing his second (mesh) law by a postulate based on conservation of energy. For a network G with m nodes and n branches $\bar{G}$ denotes the set of all oriented branches. A complex-valued function f with domain $\bar{G}$ is antisymmetrical (a.s.) if $f(X_r)=-f(X)$, where X is any oriented branch and X_r its reverse; g, defined on $\bar{G}\times\bar{G}$ is a.s. if $g(X_r, Y)=g(X, Y_r)=-g(X, Y)$. A network is defined as an ordered triple $N(G, E, Z)$ where G is a linear graph no branch of which has coincident end points, E (emf) is an a.s. function on $\bar{G}$ and Z (impedance) an a.s. function on $\bar{G}\times\bar{G}$. A basis on G denotes an ordered pair (P, B) with P a sequential ordering of the m nodes $P_1 P_2\cdots P_m$ and B a sequential ordering of n oriented branches $B_1, B_2, \cdots, B_n$, no branch being the reverse of another. Once a basis has been prescribed, E is a column matrix $E(B_k)$ and Z an $n\times n$ matrix $Z(B_j, B_k)$. A basis determines an orientation or node matrix F with $F_{jk}=-1, 1, 0$ according as P_j is the initial point of B_k, the terminal point of B_k, or not an end point of B_k. The linear network problem is defined as follows: given $N(G, E, Z)$ and some basis β, determine V and I (both a.s.) on $\bar{G}$ to satisfy (1) $V=E-ZI$, (2) $FI=0$, (3) $V^*I=0$ for all I satisfying (2) (* denotes transpose). Here (2) is Kirchhoff's first law, and (3) is shown to be the law of energy. Although a basis β is required to define the network problem, it is shown that the definition is independent of β. An algebraic argument, inspired by geometry in an n-dimensional complex Euclidean space, leads to the basic network formula without bringing in explicitly the concept of a loop. Finally loops are introduced and the equivalence of (3) with Kirchhoff's second law established.

J. L. Synge (Dublin).

Arsove, Maynard G. The algebraic theory of linear transmission networks. I, II. J. Franklin Inst. **255**, 301–318, 427–444 (1953).

Linear networks are considered having $n+1$ input terminals and $n+1$ output terminals operated at a fixed frequency. The currents and potentials at n of the input terminals constitute a $2n$-dimensional input vector. The network acts as a transformation matrix with complex matrix elements to yield a corresponding output vector. Connecting the output of one such network to the input of another corresponds to multiplication of the associated matrices. The iterative and characteristic impedance are defined in analogy with the well known quadripole case ($n=1$). A necessary and sufficient condition is stated for the existence of a characteristic impedance. The proof depends on the theory of the square root of a matrix. Various specializations are analyzed in detail. The paper closes with a treatment of the n-wire transmission line from the point of view of continuous semi-groups. [A somewhat similar theory for such generalized quadripoles was given by L. Collet, Ann. Télécommun. **4**, 42–56 (1949).]

R. J. Duffin (Pittsburgh, Pa.).

Zadeh, L. A. Optimum nonlinear filters. J. Appl. Phys. **24**, 396–404 (1953).

Classes of nonlinear filters, are considered such that each class includes the previous class. Let $u(t)$ be the filter input and let $v(t)$ be the filter output as a function of the time t. The first class is taken to be of the form $v(t)=\int_0^\infty K(u(t-x), x)dx$, the second class of the form $v(t)=\int_0^\infty\int_0^\infty K(u(t-x), u(t-y), x, y)dxdy$, etc. The functions K are taken to be arbitrary. In particular, they can involve the Dirac delta function so that v may be simply a function of u. Moreover, such filters include the class of linear filters considered by Wiener. It is shown that the optimization of such a filter in the least-squares sense leads to a linear integral equation on K. Methods for approximate realization of such filters by delay lines and linear networks in conjunction with simple nonlinear elements are indicated.

R. J. Duffin (Pittsburgh, Pa.).

Zadeh, L. A. A contribution to the theory of nonlinear systems. J. Franklin Inst. **255**, 387–408 (1953).

This paper considers the class of nonlinear two-poles described in the above review together with certain generalizations and specializations. *R. J. Duffin.*

Zadeh, L. A. Nonlinear multipoles. Proc. Nat. Acad. Sci. U. S. A. **39**, 274–280 (1953).

A multipole is defined as an operator which transforms a set of functions of the time (inputs) into another set of functions of the time (outputs). In particular an integral operator is proposed which is the vectorial generalization of the type mentioned in the second preceding review. Electrical realizations of such multipoles are discussed.

R. J. Duffin (Pittsburgh, Pa.).

Kron, Gabriel. Tensorial analysis of integrated transmission systems. I. Trans. Amer. Inst. Elec. Engrs. **70**, 1239–1248 (1951).

Tensor algebra is applied to the calculation of total and incremental power losses in the steady-state operation of interconnected electric transmission systems. The losses L are to be expressed as linear combinations of the power P supplied by each generator. In matrix notation $\mathbf{L}=\mathbf{MP}$, where $\mathbf{M}$ is a "loss" matrix, characteristic of the transmission system which depends on an average system state but is independent of $\mathbf{L}$ and $\mathbf{P}$. The calculation of the elements of $\mathbf{M}$ is based primarily on the novel idea of replacing the true load currents by a single hypothetical load current, the power remaining invariant. The transformation matrix thus obtained when allowed to operate on the system functions eventually yields $\mathbf{M}$. *R. Kahal.*

***Cerrillo, M. V., and Guillemin, E. A. Rational fraction expansions for network functions.** Proceedings of the Symposium on Modern Network Synthesis, New York, April, 1952, pp. 84–127. Polytechnic Institute of Brooklyn, New York, N. Y., 1952.

Let $f(t)$ be a real function, let $F(s)$ be its Laplace transform, and let U be the real part of F. It is pointed out that both f and F may be represented by integral formulas involving U along a suitable contour in the upper half-plane. In particular, this contour could be the positive imaginary axis. Let ϕ denote the integral of U along this axis. Then the above mentioned formulas become Stieltjes integrals with ϕ as the distribution function. This representation is more general. If ϕ is a non-decreasing function then F is a positive real function. If ϕ is of bounded variation F is termed a transfer function. Let ϕ_n be a step function with a finite number of jumps; then F_n is a rational function. Rational approximations to a given F are obtained by choosing a step function ϕ_n which approximates ϕ. Various properties of the rational approximations are considered which are pertinent to the network synthesis problem.

R. J. Duffin (Pittsburgh, Pa.).

***Darlington, Sidney. Network synthesis using Tchebycheff polynomial series.** Proceedings of the Symposium on Modern Network Synthesis, New York, April, 1952, pp. 128–139. Polytechnic Institute of Brooklyn, New York, N. Y., 1952.

This is a condensed version of a paper of the same title by the author which appeared in the Bell System Tech. J. 31, 613–665 (1952); these Rev. 14, 434. *R. J. Duffin.*

Atiya, Fuad Surial. Theorie der maximal-geebneten und quasi-Tschebyscheffschen Filter. Arch. Elektr. Übertragung 7, 441–450 (1953).

The synthesis of a four-pole with prescribed insertion-loss is given making use of methods of Darlington [J. Math. Physics 18, 257–353 (1939); these Rev. 1, 275]. The synthesis is carried out in detail for maximally-flat and quasi-Chebyshev low-pass filters. *R. J. Duffin.*

***Tellegen, B. D. H. Synthesis of four-poles.** Proceedings of the Symposium on Modern Network Synthesis, New York, April, 1952, pp. 40–49. Polytechnic Institute of Brooklyn, New York, N. Y., 1952.

Tellegen, B. D. H. Synthesis of $2n$-poles by networks containing the minimum number of elements. J. Math. Physics 32, 1–18 (1953).

The classical paper of Brune [same J. 10, 191–236 (1931)] presented the synthesis of the finite passive 2-pole network. This synthesis utilized the minimum number of reactances (capacitors and inductors) and also the minimum number of elements (reactances, resistances, and ideal transformers). The synthesis in this paper is an extension of the Brune process to a network with n pairs of terminals. Suppose that p is the order of the differential equation of free oscillations of the system composed of the $2n$-pole with its terminal pairs connected to the terminal pairs of a $2n$-pole containing no reactances. Then it is shown that the minimum number of reactances is p and the minimum number of elements is $(p+\frac{1}{2}n)(n+1)$. This paper and that of B. McMillan [Bell System Tech. J. 31, 217–279, 541–600 (1952); these Rev. 14, 116] were prepared independently but concern essentially the same synthesis problem. *R. J. Duffin* (Pittsburgh, Pa.).

Ozaki, Hiroshi. Synthèse de $2n$-pôle, étant données, dans quelque largeur de bande de fréquence, leur valeurs de matrices d'impédance et son application à la construction de transformateur d'impédances. Tech. Rep. Osaka Univ. 2, 165–175 (1952).

Suppose that the impedance matrix of a passive network with two pairs of terminals is given over a certain interval of frequencies. The problem considered is to design a network which will approximate the given one in the given interval. The simplified problem in which only the transfer function is specified was solved by R. M. Redheffer [J. Math. Physics 28, 140–147 (1949); these Rev. 11, 27]. Redheffer used a transformation of Wiener to reduce the problem to a question of trigonometric approximation. The author modifies the method of Redheffer so as to insure that the impedance matrix is of positive real type.

R. J. Duffin (Pittsburgh, Pa.).

Stöhr, Alfred. Über gewisse nicht notwendig lineare $(n+1)$-Pole. Arch. Elektr. Übertragung 7, 546–548 (1953).

Direct current multipoles are considered. The variables are the $M+1$ potentials ϕ and the $M+1$ currents i at the terminals. Properties of the linear forms associated with an infinitesimal change in the operating point are considered. If $\sum \phi di$ or $\sum id\phi$ is a total differential then the multipole is "symmetric" and conversely. Properties of input-output multipoles are related to symplectic matrices and canonical transformations of analytical mechanics. *R. J. Duffin.*

***Bayard, M. Synthesis of n-terminal pair networks.** Proceedings of the Symposium on Modern Network Synthesis, New York, April, 1952, pp. 66–83. Polytechnic Institute of Brooklyn, New York, N. Y., 1952.

Bayard, Marcel. La synthèse des réseaux à n paires de bornes. Ann. Télécommun. 7, 517–524 (1952).

One of the most interesting problems of network theory is the synthesis of a finite passive network with n pairs of terminals. In this paper the author relates his own contributions to this problem to the work of G. Gewertz, W. Cauer, Y. Oono, R. Leroy, L. Collet, and V. Belevitch. The impedance matrix Z of a network is an n by n matrix whose matrix elements are rational functions of the complex frequency p. The first synthesis method described consists of first removing the poles on the imaginary axis. Then $Z=Z''+Z'$ where Z'' and Z' are even and odd functions of p respectively. The passivity condition demands that the quadratic form associated with Z'' be positive semi-definite for p or the imaginary axis. By completing the square this quadratic form may be reduced to a sum of squares. It is thereby seen that Z may be expressed as a sum of matrices of the form Z_1. Here Z_1 is an impedance matrix whose even part is of rank 1. The determinant of Z_1 vanishes for $p=0$. Thus Z_1^{-1} has a pole at the origin and the reduction process may be repeated. If Z_1 does not have an inverse it is shown that $Z_1=AZ_2\bar{A}$ where A is a real constant matrix and Z_2 is an $n-1$ by $n-1$ impedance matrix. The same reduction process can then be applied to Z_2. Another method described employs a purely reactive network with $2n$ pairs of terminals, n of the terminal pairs corresponding to Z and the others terminated by resistors. The proof consists in verifying that the impedance matrix X of the network with $2n$ pairs of terminals satisfies the purely reactive conditions. Methods for the synthesis of reactive networks are well known. The paper is concluded by a discussion of the scattering matrix $S=(I-Z)(I+Z)^{-1}$ introduced by Belevitch. It is stated that methods of Belevitch seems to give the most general solution of the synthesis problem. [Another synthesis method, not described in this paper, is given by McMillan, Bell System Tech. J. 31, 217–279, 541–600 (1952); these Rev. 14, 116.] *R. J. Duffin.*

***Giordano, A. B. Driving point impedances.** Proceedings of the Symposium on Modern Network Synthesis, New York, April, 1952, pp. 21–39. Polytechnic Institute of Brooklyn, New York, N. Y., 1952.

***Fialkow, A. D. Two terminal-pair networks containing two kinds of elements only.** Proceedings of the Symposium on Modern Network Synthesis, New York, April, 1952, pp. 50–65. Polytechnic Institute of Brooklyn, New York, N. Y., 1952.

Seal, Philip M. Analysis of network response to periodic waves. J. Franklin Inst. 257, 13–24 (1954).

Quantum Mechanics

Corinaldesi, E. Some aspects of the problem of measurability in quantum electrodynamics. Nuovo Cimento (9) **10**, supplemento, 83–100 (1953).

Previously L. Landau and R. Peierls [Z. Physik **69**, 56–69 (1931)] had found that there existed a limitation on the accuracy with which an electromagnetic field could be measured, beyond what would be expected from the quantum theory of the field. However, N. Bohr and L. Rosenfeld [Danske Vid. Selsk. Mat.-Fys. Medd. **12**, no. 8 (1933)] pointed out that this result was due to the atomic nature of the test particle assumed to be used and could be avoided by the use of test bodies of small but finite extension. In the present paper the author investigates the question of the measurement of the field from the standpoint of quantum electrodynamics and finds that the accuracy is limited by the fluctuations of the field, which tend to infinity as the volume of the region and the time interval of measurement tend to zero. Taking into account the interaction of the field with matter requires the introduction of renormalization. In this case it is found that even for a finite volume, if the time goes to zero, the mean square deviation of an electric field component diverges. The measurement of charge and current density is also discussed.

N. Rosen (Haifa).

Ludwig, G. Der Messprozess. Z. Physik **135**, 483–511 (1953).

This paper makes a careful examination, within the general framework of von Neumann's Mathematische Grundlagen der Quantenmechanik [Springer, Berlin, 1932], of the mathematical formulation of the process of measurement in quantum theory. If I is a physical system which is to be measured by means of the physical system II, the apparent conflict between the macroscopic character of II and the thermodynamic irreversibility of motion on the one hand, and the quantum-mechanical character of I and II and the unitary, reversible form of motion in quantum mechanics, on the other hand, is formulated and resolved along mathematical lines, under special assumptions about the character of the Hamiltonian as an operator on Hilbert space. It is further indicated how a macroscopic physics (in particular, classical mechanics and thermodynamics) can be derived from a quantum physics, the main point being that as far as the macroscopic observables of the quantum system are concerned, the system can be described in classical terms. Relevant ergodic theorems are obtained.

I. E. Segal (New York, N. Y.).

Schönberg, M. A statistical generalization of the quantum mechanics. I. Nuovo Cimento (9) **10**, 1499–1538 (1953).

A detailed discussion is given for the generalization of quantum mechanics previously proposed by the author [Nuovo Cimento (9) **10**, 350–353 (1953); these Rev. **14**, 825]. *L. Van Hove* (Princeton, N. J.).

Schönberg, M. Generalization of the classical field formalism by means of functionals. Nuovo Cimento (9) **10**, 1597–1601 (1953).

The paper proposes the application to general fields of the functional formalism used before by the author to generalize quantum mechanics [Nuovo Cimento (9) **10**, 350–353 (1953); these Rev. **14**, 825; see also the preceding review] and treats the example of the Klein-Gordon field equation. *L. Van Hove* (Princeton, N. J.).

Castoldi, Luigi. Operatori di stato nella meccanica quantica degli insiemi di particelle. Introduzione alla teoria della seconda quantizzazione. Atti Accad. Ligure **9** (1952), 113–135 (1953).

An elementary, largely conventional, treatment is given of the second quantization, with emphasis on the statistical aspects, and notably the connection with von Neumann's density matrices. *I. E. Segal* (New York, N. Y.).

Heisenberg, Werner. Doubts and hopes in quantum-electrodynamics. Physica **19**, 897–908 (1953).

Rosenfeld, L. Problems of interpretation of quantum electrodynamics. Physica **19**, 859–868 (1953).

Dirac, P. A. M. Les transformations de jauge en électrodynamique. Ann. Inst. H. Poincaré **13**, 1–42 (1952).

An exposition of the results of two previous papers [Canadian J. Math. **2**, 129–148 (1950); Nuovo Cimento (9) **7**, 925–938 (1950); these Rev. **13**, 306]. The theories of a spinless charged particle and of a Dirac electron are formulated in the Poisson bracket notation without recourse to Fourier analysis of the fields and without restricting the allowable gauge transformations. *A. J. Coleman.*

Ferretti, B. Sulla diagonalizzazione della hamiltoniana nella teoria dei campi. II. Nuovo Cimento (9) **10**, 1079–1125 (1953).

In an earlier paper [Nuovo Cimento (9) **8**, 108–131 (1951); these Rev. **12**, 890] the author outlined his program for calculating the solutions of the equations of motion of a quantum-mechanical system with Hamiltonian $H=H_0+\lambda V$. The idea is to avoid expanding in powers of the "coupling constant" λ, and instead to find the differential equations which physically observable quantities satisfy as functions (not necessarily analytic) of λ. For example, given an observable quantity A depending on λ, the program consists of calculating $A(\lambda+\delta\lambda)$ in terms of $A(\lambda)$ and other calculable quantities, to first order in $\delta\lambda$.

In this paper the program is applied to quantum electrodynamics, in which V is the interaction between the quantized Maxwell and Dirac fields, and λ is the charge of the electron. In this case the power-series in λ probably diverges. On the other hand, the removal of self-energy and self-charge infinities by renormalization has been previously carried through only for the power-series solutions. The author claims in this paper to carry through the renormalization completely without power-series. In view of the importance of the problem, one must examine carefully how far his claim is justified by what he has achieved.

Suppose $A(\lambda)$ to be the matrix element of a given operator between given "free particle" states in the Feynman representation. Then $A(\lambda)$ is formally a sum of an infinite number of contributions $A(\lambda, G)$ from different Feynman graphs G with a fixed set of external lines. The sum $\sum_G A(\lambda, G)$ is the power-series in λ which we wish to avoid. For each G we have

$$(1) \qquad A(\lambda+\delta\lambda, G)=A(\lambda, G)+\sum_B B(\lambda, G')\delta\lambda.$$

Here the sum $\sum_B$ is over a finite number of terms which are to be calculated from a finite set of graphs G'. Each G' is obtained from G by adding one vertex in a particular way. The divergences arising in each $B(\lambda, G')$ can be renormalized by the usual methods.

The author's essential idea is that the graphs G and G' can be divided into a finite number of "types". The type of a graph is determined by stating which subsets of its external lines can be separated from the remaining external lines by cutting one, two, or three of the internal lines. Since for a given $A(\lambda)$ the external lines of every G are the same, there is only a finite number of types to be considered. Furthermore, the detailed method of separating and eliminating divergences from a graph is identical for all graphs of a given type. Therefore, summing (1) over all graphs gives

$$A(\lambda+\delta\lambda)=A(\lambda)+\sum_{T}\sum_{B,G'}B(\lambda,G')\delta\lambda. \tag{2}$$

Here $\sum_T$ is a sum over a finite number of types. And the inner sum, although infinite, is a sum over graphs of the same type. The renormalization of the right side of (2) can be carried out formally in a finite number of steps, if it is assumed that all quantities $B(\lambda)$ are known and finite.

Summarizing, the author has proved that the renormalization of $A(\lambda+\delta\lambda)$ can be performed to first order in $\delta\lambda$, if previously all the quantities $B(\lambda)$ have been renormalized.

The question remains open, whether this renormalization of the differential equation for $A(\lambda)$ will actually enable $A(\lambda)$ to be calculated for a finite λ. There is in fact an infinite set of quantities $A(\lambda), B(\lambda), \cdots$, and the differential equations are an infinite set of coupled equations giving $(dA/d\lambda), (dB/d\lambda), \cdots$ in terms of $A, B, \cdots$ in a complicated way. Since each $A, B, \cdots$ is a function of a set of free-particle momenta, the equations are integro-differential equations. The feasibility of the renormalization program requires not only that the equations have a solution, but also that the functions $A(\lambda), B(\lambda), \cdots$ shall for each λ be sufficiently rapidly decreasing functions of the particle momenta. Since a general theory of infinite systems of integro-differential equations does not exist, the author's program can only be carried through by a detailed discussion of the behavior of the equations to which he is led.

F. J. Dyson (Princeton, N. J.).

Schwinger, Julian. The theory of quantized fields. IV. Physical Rev. (2) **92**, 1283–1299 (1953).

This is the fourth paper of a series [Physical Rev. (2) **82**, 914–927 (1951); **91**, 713–728, 728–740 (1953); these Rev. **13**, 520; **15**, 81; and earlier papers cited there]. It applies the author's general formalism to the construction of the quantized field theory of a Dirac field interacting with a given external spinor source-function. The results are throughout the same as those obtained by older methods. The principal novelty in this paper is the formal definition of an "eigenvector" Ψ of a Dirac field quantity $\chi(x)$. The eigenvector must satisfy the equations $[\chi(x)-\chi'(x)]\Psi=0$, where $\chi'(x)$ is a function of the space-point x with the property that $\chi'(x)$ anti-commutes with every $\chi'(y)$ and with every Dirac field-operator $\chi(y)$. The $\chi'(x)$ are not of course ordinary numbers, but they can be considered in a formal way as "eigenvalues" of the $\chi(x)$. *F. J. Dyson.*

Brueckner, K. A., and Watson, K. M. The construction of potentials in quantum field theory. Physical Rev. (2) **90**, 699–708 (1953).

An inductive method of obtaining the interaction potential between two particles in field theories of various types is based on Lippmann and Schwinger's time-independent formulation of the scattering problem [Physical Rev. (2) **79**, 469–480 (1950); these Rev. **12**, 570]. For handling this method algebraic techniques are developed under the assumption that the interaction Hamiltonian is a homogeneous linear polynomial in the field variables. For the interaction potential a series expansion is obtained where the nth term is expressed in terms of the preceding $(n-1)$ terms. In contrast to the Tamm-Dancoff and Bethe-Salpeter methods this series is not a power series in the coupling constant g; its terms have the form of fractional expressions where the coupling constant occurs also in the denominator. Thus, if the coupling becomes large, successive terms may not increase with g as rapidly as in the usual power series expansions. The structure of these terms is relatively simple compared, for instance, to the complexity of the Feynman diagrams for high-order processes. The simplification results from considering explicitly the "scattering" of virtual particles before reabsorption. The method is applied to a linear meson theory without nucleon pair production, to a general nonlinear theory without pair production, and finally to a general nonlinear theory with pair production. Simple illustrative examples are given for the linear and nonlinear theories. *E. Gora* (Providence, R. I.).

Karpman, V. I. On a connection between the method of regularization and theories of particles with arbitrary spin. Doklady Akad. Nauk SSSR (N.S.) **89**, 257–260 (1953). (Russian)

It is shown that it is not possible to identify the auxiliary masses used in the regularization method of Pauli and Villars [Rev. Modern Physics **21**, 434–444 (1949); these Rev. **11**, 301] with the masses of particles of arbitrary spin described by equations of the type $(L^{\lambda}\nabla_{\lambda}+i\kappa)\psi=0$.

E. Gora (Providence, R. I.).

Gellikman, B. T. On the quantum theory of wave fields. Doklady Akad. Nauk SSSR (N.S.) **90**, 359–362 (1953). (Russian)

A treatment of the case of strong coupling between scalar and pseudoscalar meson fields and nucleons ($g^2/hc\gg 1$) is based on the assumption that the wave function of the field can be represented as the sum of a large classical component and of a small operator component. The nucleus is described as an extended source of "radius" a. The classical component of the wave function is obtained first, and used as starting point in an approximation procedure for determining the operator component. The method is applicable if the operator component is much smaller than the classical component, and this is shown to be the case for meson energies much smaller than g^2/a only. *E. Gora.*

Utiyama, Ryôyû. On the convergence of the perturbation method in the quantum field theory. Progress Theoret. Physics **9**, 593–606 (1953).

The author with T. Imamura [same journal **9**, 431–454 (1953); these Rev. **15**, 82] has proved the divergence of perturbation theory expansions for the case of a quantized scalar field with a nonlinear and nonlocal interaction. In this paper it is shown that the expansions converge if a certain term is dropped from the integral equations defining the Green's functions for the field. Dropping this term has the effect of enormously reducing the number of Feynman graphs which contribute to the perturbation expansion. The author proposes that the simplified equations be considered seriously as a basis for a divergence-free field theory. He does not investigate the physical consequences of this proposal. *F. J. Dyson* (Princeton, N. J.).

Källén, Gunnar. Non perturbation theory approach to renormalization technique. Physica 19, 850–858 (1953).

This is a very clear and non-technical account, given at the Leiden conference in June 1953, of the author's new treatment of quantum electrodynamics. The mathematical details have been published in earlier papers [Helvetica Phys. Acta 25, 417–434 (1952); Danske Vid. Selsk. Mat.-Fys. Medd. 27, no. 12 (1953); these Rev. 14, 435; 15, 79]. This paper includes some discussion which took place at the end of the author's lecture. *F. J. Dyson.*

Kalitzin, Nikola St. Über eine neue Kerntheorie. Acta Phys. Acad. Sci. Hungar. 3, 45–53 (1953). (Russian summary)

This paper is a continuation of the work of Kalitzin [Izvestiya Bulgar. Akad. Nauk. Otd. Fiz.-Mat. Tehn. Nauk. Ser. Fiz. 2, 63–78 (1952); these Rev. 15, 169] and of Zaycoff [ibid. 2, 79–98 (1952); these Rev. 15, 170] based on a space of six dimensions and dealing with field equations involving a tensor potential. The author arrives at quantities which he interprets as the spin and two charges of a nucleon. In the static case he obtains an expression for the interaction between two nucleons involving these quantities and finds that the spin-interaction term is free from the usual dipole-potential singularity. *N. Rosen* (Haifa).

Miyatake, Osamu. On the singularity of the perturbation-term in the field quantum mechanics. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. 3, 145–155 (1952).

The hamiltonian H of an electron in an electromagnetic field is split, in two ways, into the sum of an unperturbed and a perturbing term, H_0 and H' respectively. In one, H_0 is the sum of the free electron and radiation fields; in the other, it is the Bloch and Nordsieck term [Physical Rev. (2) 52, 54–59 (1937)]. The author shows that eigenvalues of the two H_0's are orthogonal, and that their corresponding zero energies differ by infinity. The other main result is identical with that of a paper of Van Hove [Physica 18, 145–159 (1952); these Rev. 14, 118] to which no reference is made, namely, the eigenstates of the total hamiltonian of a fixed nucleon in interaction with a neutral meson field are orthogonal to the eigenstates for the same system without interaction. Therefore the complete system cannot be represented by combinations of the free-field states.

A. J. Coleman (Toronto, Ont.).

Fabre de la Ripelle, Michel. Résolution des équations de perturbation. III. Expression des constantes de transition. C. R. Acad. Sci. Paris 237, 874–876 (1953).

An extension of previous work [same C. R. 236, 459–461 (1953); these Rev. 14, 708] on perturbation theory in which one computes the probability of finding a system in a given state after going through an arbitrary number of intermediate states, using a saddle-point method of approximation.

H. C. Corben (Pittsburgh, Pa.).

Hellund, E. J., and Brachman, M. K. Space-time representation in wave mechanics. Physical Rev. (2) 92, 822–824 (1953).

The Schrödinger equation is written in a representation based on an orthonormal set of functions in space and time, each vanishing outside a cubic cell. The equation thus obtained is presented as a basis for a space-time, path-like description of quantum-mechanical motion.

L. Van Hove (Princeton, N. J.).

Green, H. S. A pre-renormalized quantum electrodynamics. Proc. Phys. Soc. Sect. A. 66, 873–880 (1953).

The conventional formulation of quantum electrodynamics [Dyson, Physical Rev. (2) 75, 1736–1755 (1949); these Rev. 11, 145] starts with "bare" particle propagation functions $G_0(x)$ and $D_0(x)$ which are finite and then proceeds to construct the propagation functions $G(x)$ and $D(x)$ for real particles. These latter are found to contain infinities which are removed by renormalization. The author's point of view is that it is the unphysical abstractions represented by G_0, D_0, and γ_0 which (should) contain the infinities, for these represent electron and photon lines without self-energy processes and vertices without radiative corrections. If the theory were rewritten such that these quantities (with their infinities) were entirely eliminated from it, the resulting (pre-renormalized) form of the theory would be "invariant against renormalization". This is accomplished by starting with Dyson's equation, in momentum space,

$$[G(p)]^{-1}=[G_0(p)]^{-1}+e^2\int\Gamma(p;k)G(p-k)D(k)\gamma_0 dk,$$

and the corresponding equation for $D(p)$. New equations are obtained from these two by a procedure analogous to functional differentiation in configuration space. These latter equations represent physical processes obtainable diagrammatically by insertion of photon vertices in the graphs for $G(p)$ and $D(p)$. Then G_0, D_0, and γ_0 can formally be eliminated between these and Dyson's equations. A mathematical procedure for generating the usual perturbation expansion forms for $G(p)$, $D(p)$ and other S-matrix elements (such that the divergences all go with the G_0, D_0, and γ_0) is indicated. *A. Salam* (Cambridge, England).

Schild, A. A new modification of classical electromagnetic theory. Physical Rev. (2) 92, 1009–1014 (1953).

Consider a single charged particle, with a variable rest mass m, in a prescribed electromagnetic field $A_\mu(x)$, and introduce a parameter u, defined by $du=ds/m$ (ds is the proper time interval) to describe its motion. The a variation of the action integral

$$\int_{u_1}^{u_2}\left[\frac{dx^\mu}{du}\frac{dx_\mu}{du}+2e\frac{dx^\mu}{du}A_\mu\right]du$$

subject to $\delta x^\mu=0$, $\delta u=0$, for $u=u_1$, u_2, gives (1) the usual Lorentz equations, as the Eulerian equations; (2) $m^2=$constant as the Jacobian integral of motion. The constancy of rest mass, although not assumed a priori, thus follows as a consequence of the equations of motion. If A_μ is a function of the momenta as well, the Jacobian integral no longer reduces to $m^2=$constant and the rest mass of the particle changes during its motion.

Using this simple idea, the author (1) gives a new action principle for classical electrodynamics to replace Fokker's, the constancy of rest mass coming out as a consequence of the equations of motion, as above; (2) considers a further modification of the new action principle, on the lines of Feynman's modification [Physical Rev. (2) 74, 939–946 (1948); these Rev. 10, 22?] of Fokker principle, where finite self-actions for the particles are obtained by replacing Fokker's δ-functions by other approximating functions f. The author uses form-functions f which depend on momenta as well, thereby avoiding the difficulty Feynman encountered of obtaining the same electromagnetic rest mass for all types of charged particles (electrons, protons, etc.) once f was fixed. In this new modification of electrody-

namics, particles interacting at small distances no longer have constant rest masses. *A. Salam.*

Hamilton, J. The Fredholm theory of the S matrix. Physical Rev. (2) **91**, 1524–1526 (1953).

The Fredholm theory of linear integral equations has been used by Salam and Matthews [Physical Rev. (2) **90**, 690–695 (1953); these Rev. **15**, 82] to study the convergence of the Feynman-Dyson solution to problems involving a quantised spinor field in interaction with a prescribed time-dependent electromagnetic field. They could thus show that the solution would converge for all strengths of the prescribed field.

In this paper, the author has applied the Fredholm formalism to the integral equation satisfied by the (interaction representation) transformation operator, for the completely general case of two interacting quantised fields. The integral equation here (in contrast to the equation discussed by Matthews and Salam) is a q-number equation, involving as its kernel the (possibly unbounded) interaction Hamiltonian operator. As the author points out, the Fredholm formalism need thus have no a priori meaning. It has, however, been possible to draw one conclusion. By writing down explicitly the expression for the Fredholm determinant, it is shown that this determinant can be written (formally) as an exponential of a pure imaginary quantity. It would thus have no "zeros" to cancel the "poles" of the iteration solution, as in the conventional theory, and no extension of the region of convergence would seem possible. *A. Salam* (Cambridge, England).

Watanabe, Satosi. Reversibility of quantum electrodynamics. Physical Rev. (2) **84**, 1008–1025 (1951).

The author discusses the meaning of, and some of the consequences of, time reversal in quantum electrodynamics. He points out that there are two possible definitions:

$$\begin{array}{lllll} & \mathbf{x}\to\mathbf{x} & \mathbf{v}\to-\mathbf{v} & & \\ (1) & e\to e, & \mathbf{E}\to\mathbf{E}, & \mathbf{H}\to-\mathbf{H} & \\ (2) & e\to -e, & \mathbf{E}\to-\mathbf{E}, & \mathbf{H}\to\mathbf{H} & \end{array}$$

This ambiguity precludes a unique connection between charge conjugation and time reversibility. If definition (1) is used, reversibility alone does not distinguish between Fermi-Dirac or Einstein-Bose statistics for either the charged spinor or scalar field. Of interest mathematically is his introduction of such quantities as pseudo-spinors in order to conveniently discuss time reversal. He also uses an equivalent representation in which the usual four-component Dirac spinor is replaced by an eight-component variety. *H. Feshbach* (Cambridge, Mass.).

Drukarev, G. F. On the theory of the passage of particles through a potential barrier. Akad. Nauk. SSSR. Žurnal Eksper. Teoret. Fiz. **21**, 59–68 (1951). (Russian).

For the case of a spherically symmetric potential barrier the author considers the Schroedinger wave function as describing a non-stationary process. Taking the wave function as initially localized within the potential barrier, he investigates its behavior outside the barrier. He finds that if the particle is initially in a quasi-stationary state then, in the case of interest, its absolute value outside the barrier after a long time will increase exponentially with distance for $r<vt$ and will fall off rapidly for $r>vt$, where v is the speed of the particle. In the general case, the wave function can be written as a sum of terms corresponding to the disintegration of various quasi-stationary states and terms describing the spreading out of a wave packet.

It should be pointed out that, strictly speaking, the assumption that the particle is in a quasi-stationary state is, in general, inconsistent with the assumed initial localization of the wave function inside the barrier. *N. Rosen.*

Takasu, Tsurusaburo. A combined field theory as a three-dimensional non-holonomic parabolic Lie geometry and its quantum mechanics. Yokohama Math. J. **1**, 105–116 (1953).

In a former publication [same J. **1**, 1–28, 29–38, 39–74, 75–77, 79–82, 83–87 (1953); these Rev. **15**, 350] the author introduced in every space of linear connection (not invariant) curves which are as for meet and join as straight lines in affine space. This enables him to interpret the combined field theory as a "non-holonomic" parabolic Lie geometry. *J. A. Schouten* (Epe).

Costa de Beauregard, Olivier. Intégrales de Fourier covariantes et résolution du problème de Cauchy pour les particules libres de spin 1/2 et 1. C. R. Acad. Sci. Paris **237**, 1495–1497 (1953).

Curtiss, C. F. The separation of the rotational coordinates from the N-particle Schroedinger equation. II. J. Chem. Phys. **21**, 1199–1208 (1953).

[For part I see Curtiss, Hirschfelder and Adler, same J. **18**, 1638–1642 (1950); these Rev. **12**, 782.] The Schroedinger equation is considered for a system of particles regarded as consisting of two sets, so that one might think of the particles as atoms and the two sets as two molecules. In place of the coordinates of the individual particles new coordinates are introduced: the coordinates of the center of mass of the system, the relative coordinates of the center of mass of one molecule with respect to that of the other molecule and for each molecule the relative coordinates of the particles with respect to its center of mass. The relative coordinates are expressed in terms of rotational coordinates, which are separated from the Schroedinger equation by the use of group-theoretical methods. The coordinates finally used to describe the configuration of the particles in a center-of-mass system are those describing the orientation of the system as a whole, the relative orientation and the separation of the molecules, and the internal configuration of each molecule. Only the coordinates describing the separation of the molecules and their internal configurations are left in the Schroedinger equation, which now takes the form of a set of coupled differential equations for functions of these coordinates. The equations are applied to a system consisting of two diatomic molecules, which represent a special case. *N. Rosen* (Haifa).

Glauber, R. J. On the gauge invariance of the neutral vector meson theory. Progress Theoret. Physics **9**, 295–298 (1953).

Neutral vector meson theory is developed in terms of five field variables as suggested by Stückelberg [Helvetica Phys. Acta **11**, 225–244 (1938)]. In addition to obeying a usual supplementary condition the theory is required to be invariant under gauge transformations. The relation of the theory to its limiting case, electrodynamics, is considered. *C. Strachan* (Aberdeen).

Petiau, Gérard. Sur une nouvelle théorie des corpuscules de spin quelconque et son application au calcul des sections efficaces de diffusion coulombienne. J. Phys. Radium (8) **14**, 501–509 (1953).

Dirac's free-particle equation is generalized to a wave function of 4^n components (n a positive integer) the corresponding Dirac operators being representable in terms of n systems of the usual operators. It is shown that such an equation describes a particle of spin $nh/2$, the rest-mass then possessing the values $(n/|n-2p|)m_0$ where m_0 is a constant and $p=0, 1, \cdots, n$, $2p\neq n$. The general properties of the equation are considered for the cases n even and n odd, and the Møller scattering cross-section is evaluated for two such particles. *H. C. Corben* (Pittsburgh, Pa.).

Payne, W. T. A geometric approach to nonrelativistic spin theory. Amer. J. Phys. **21**, 621–628 (1953).

Roger, Frédéric. Sur une image corpusculaire et un prolongement matriciel de la méthode ondulatoire des orbitales moléculaires. C. R. Acad. Sci. Paris **236**, 2207–2209 (1953).

A formal similarity between a classical problem of vibrations and a secular equation of quantum mechanics for molecular orbitals is mentioned. Use will be made of this later. *C. Strachan* (Aberdeen).

Chraplyvy, Zeno V. Reduction of relativistic two-particle wave equations to approximate forms. I. Physical Rev. (2) **91**, 388–391 (1953).

This paper is based on the work of Foldy and Wouthuysen [Physical Rev. (2) **78**, 29–36 (1950)] in which, by means of a canonical transformation on the Dirac Hamiltonian, the Dirac equation for a single particle is put into a form such that positive and negative energy states are represented separately by two-component wave functions. In the present work this procedure is extended to a two-particle equation. The method is applied to the Breit equation [ibid. **34**, 553–573 (1929)], which involves a sixteen-component wave function, and one obtains separate equations for different combinations of signs of the energies of the two particles. The equation for the states in which both energies (or masses) are positive, involving a four-component wave function, is compared with the approximate four-component Breit equation obtained by elimination of the smaller components. The method under discussion fails if the rest masses of the particles are equal. The author examines the conditions that would have to be satisfied in the case of equal masses and concludes that no finite transformation exists which would convert the two-particle Hamiltonian into the desired form. *N. Rosen* (Haifa).

Riddell, Robert J., Jr. The number of Feynman diagrams. Physical Rev. (2) **91**, 1243–1248 (1953).

Using simple combinatorial and algebraic arguments, the author constructs the generating power-series whose coefficients represent the number of Feynman diagrams with a given number of vertices. The results agree with those of C. A. Hurst [Proc. Roy. Soc. London. Ser. A. **214**, 44–61 (1952); these Rev. **14**, 607]. *F. J. Dyson.*

Durand, Emile. Le principe de Huygens et la diffraction de l'électron en théorie de Dirac. C. R. Acad. Sci. Paris **236**, 1337–1339 (1953).

The author derives an expression relating the value of a Dirac wave function at a point in terms of its values on a hypersurface in four space. *H. Feshbach.*

Aeschlimann, F. Sur la représentation géométrique des corpuscules et la méthode triondulatoire. J. Phys. Radium (8) **13**, 600–604 (1952).

Destouches-Février, P. Les prévisions et la méthode triondulatoire. J. Phys. Radium (8) **13**, 605–609 (1952).

It is proposed that the usual wave function $\psi(M, t)$ be generalized by replacing the point M by a function u of space-time, so that ψ becomes a functional $X(u, t)$. It is shown that the generalization is feasible, and advantages of the generalization are indicated. *C. C. Torrance.*

Eder, Gernot. Statistik virtueller Mesonen. Acta Physica Austriaca **7**, 278–282 (1953).

The author proposes to modify the meson-theoretical calculation of nuclear forces according to two assumptions: the contribution of each plane wave component of the meson field is equal to the second-order perturbation value multiplied by the average number of mesons present in the plane-wave state; this average number corresponds to a thermal distribution of (a given number of) virtual mesons over all plane-wave states. Self-energies are thus found finite. The same idea extends to the photon field and to fermion fields. Its physical applicability is not discussed. *L. Van Hove.*

Yukawa, Hideki. Structure and mass spectrum of elementary particles. I. General considerations. Physical Rev. (2) **91**, 415–416 (1953).

Yukawa, Hideki. Structure and mass spectrum of elementary particles. II. Oscillator model. Physical Rev. (2) **91**, 416–417 (1953).

Mehl, Clarence R., and Havas Peter. The classical scattering of neutral mesons. Physical Rev. (2) **91**, 393–397 (1953).

The classical theory of Bhabha and of Harish-Chandra for the motion of point particles in vector and scalar meson fields respectively may be treated from the point of view of the Wheeler-Feynman theory of action at a distance. In contrast to electrodynamics the equations of motion from the action-at-a-distance theory then differ from those derived from the field theory. For the former of these theories the cross-sections for the classical scattering of scalar and vector mesons by nucleons are computed here and compared with those derived from the original field theory. *H. C. Corben* (Pittsburgh, Pa.).

Gell-Mann, M., and Goldberger, M. L. The formal theory of scattering. Physical Rev. (2) **91**, 398–408 (1953).

A discussion is given of the preparation of the state vector of a system so that it will be appropriate for the description of scattering phenomena. In terms of this it is possible to formulate a treatment of scattering which is here compared with the usual presentations. Applications are made to the theory of scattering by two potentials and to the theory of self-energies. *H. C. Corben* (Pittsburgh, Pa.).

Nagahara, Sigeru. The scattering of slow electrons by the diatomic molecules. I. General theory. J. Phys. Soc. Japan **8**, 165–168 (1953).

A general theory for the scattering of slow electrons by diatomic molecules is developed. By using ellipsoidal coordinates and expansions in so-called spheroidal wave functions, the author derives explicit expressions for the differential and the total cross-sections, but the numerical applications are reserved for a later paper. *P. Löwdin.*

PART II IS AN INDEX WHICH HAS
BEEN PHOTOGRAPHED AT THE BEGINNING
OF THE VOLUME(S).